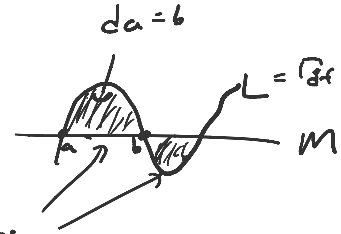
Holography and Skew Howe Duality

Friday, October 29, 2021 9:15 AM

0. Intro to the A-model

Ex: $L = \Gamma_{df} \subseteq T^*M$ $\quad M \xrightarrow{f} \mathbb{R}$ $\rightsquigarrow$

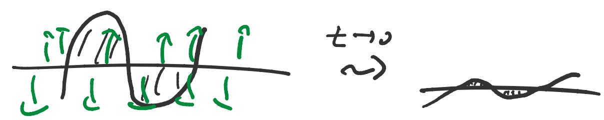


$$\mathrm{Hom}_{\mathrm{Fuk}(T^*M)}(L, M) = (\mathbb{C}\langle L\cap M\rangle, \underset{\substack{\uparrow \\ \text{count} \\ \text{hol. strips}}}{d})$$

$$= (\mathbb{C}\langle \mathrm{Crit}(f)\rangle, d_{\mathrm{Morse}}) \quad \text{Morse thy of } M$$

$\rightsquigarrow$ can model this as $(\Omega_M, d_{dR} + df \wedge -)$.

Consider family $L_t = \Gamma_{tdf}$, $t \to 0$



Idea: Action of this hol. strip is $f(b) - f(a)$. In the limit $t \to 0$, this is 0 $\rightsquigarrow$ we're computing Hom space using only constant (or 0 action) hol. disks.

↖ Can be seen perturbatively.

In genl: Suppose (X, ω) is Liouville: $\omega = d\lambda$, $\lambda = \iota_v \omega$.

If L_i are conic for the Liouville vector field V ($\Longleftrightarrow \lambda|_{L_i} = 0$), then can be no non-constant hol. disks w/ boundary on L_i.

(If $L = \Gamma_{df} \subseteq (T^*M, \omega = d\lambda)$, $\boxed{\lambda|_L = f}$)

In a Liouville manifold, study Lag. L by pulling it backwards under Liouville flow. [Nadler-Shende]: Nearby cycles map for $\mathrm{loc}(L) \to \mathrm{sh}(\text{skeleton} = \text{stable set of Liouv. flow})$

Main example: $(X, \omega) = T^*M$. Conic Lagrangians include M and $T^*_S M$, $S \subseteq M$ submfld.

We've seen that $\mathrm{End}(M) \simeq C^*(M)$.

Q: What about Homs among other conic Lagrangians?

A: Locally, [NZ, GPS], $\mathrm{Fuk}(T^*M)$ can be modeled as a category of constructible sheaves on M.

$$(\text{Conic Lagrangian}) \longleftrightarrow (\text{singular support of sheaf})$$

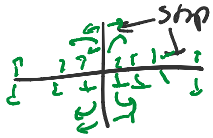
Warning: Usual "Wrapped" Fukaya cat of T^*M is not equivalent to $\mathrm{Sh}(M)$.

possible fixes:
- Don't wrap ← bad categorical properties
- Pick some particular noncompact conic Lagrangian and

possible fixes: • Don't wrap ← and categorical properties

• Pick some particular noncompact conic Lagrangian and stop wrapping them. ← no longer CY, only rel. CY (Tanaka-Shih)$^{-1}$ (Brav-Dyckerhoff)

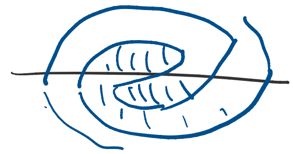
<u>Ex</u>: • $T^*[0,1]$  ← zero section ⇝ $\mathrm{Fuk}_{\mathcal{B}}$ = Loc(zero-section)

• $\mathbb{D}$ 

stop wrapping here

↖ can think of the ws subset of $Sh(\mathbb{R})$ generated by $\underline{k}_{\mathbb{R}}, \underline{k}_0$

$$\text{Wrapped Fuk}(T^*\mathbb{R}^n) = 0$$



<u>Rmk</u>: If M is mfld w/ boundary,

$$\mathrm{Fuk}(T^*M) = \mathcal{L}oc(M) := \mathrm{Perf}(C_\cdot(\Omega M))$$

not CY becuz M doesn't have Poincaré duality

(CY rel $\mathcal{L}oc(\partial M)$).

<u>Ex</u>: In T^*M,

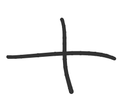
$M \leftrightarrow \underline{k}_M$

$T^*_S M \leftrightarrow \underline{k}_S$.

d-dim

$$\mathrm{End}_{\mathrm{Fuk}(T^*M)}(M \oplus T^*_S M) = \mathrm{End}_{Sh(M)}(\underline{k}_M \oplus \underline{k}_S) = \begin{pmatrix} C^\cdot(M) & C^\cdot(S) \\ C_\cdot(S)[d-n] & C^\cdot(S) \end{pmatrix}$$

$$\begin{aligned} \mathrm{Hom}(\underline{k}_S, \underline{k}_M) &= \mathrm{Hom}(\underline{k}_S, f^! k[-n]) \quad (\omega_M[n] \cong,\ f: M \to pt) \\ &= \mathrm{Hom}(f_! \underline{k}_S, k[-n]) \\ &= \mathrm{Hom}(C^\bullet_c(S), k[-n]) \\ &= C_\cdot(S)[-n+d] \end{aligned}$$

<u>Ex</u>:  $\subseteq T^*\mathbb{R}$ has end. alg $\begin{pmatrix} k & k \\ k[-1] & k \end{pmatrix}$.

<u>Ex</u>: L_1, L_2 lin. Lags in $T^*\mathbb{R}^n$ meet in an r-plane

$$\rightsquigarrow \begin{pmatrix} k & k \\ k[r-n] & k \end{pmatrix}.$$

<u>1. Holography</u>. <u>Setup: Gravitational background</u> $\mathcal{C}$ CY cat

(which is "geometric" ← local ones ⇝ $HC_-(\mathcal{C})$ is part of the field (which is "geometric" ≈ local over spacetime? X ⇝ $HC_-(\mathcal{C})$ is part of the field content of some thy on spacetime X.

Branes: N copies of object $\mathcal{F} \in \mathcal{C}$, placed along some subspace $L \subseteq X$.

Brane gauge thy is controlled by local Lie alg $\mathfrak{g}_{\text{gauge}} \sim \mathrm{End}_{\mathcal{C}}(\mathcal{F}^{\oplus N})$ living over L $= \underline{\mathfrak{gl}_N}(\mathrm{End}_{\mathcal{C}}(\mathcal{F}))$

in degree 0 ⇝ ghosts

Closed-string field thy:

$\mathfrak{g}_{\text{grav}} = \underbrace{HC_\cdot^-(\mathcal{C})}_{HH_\cdot(\mathcal{C})^{S^1}}[1-d] = HC^\cdot(\mathcal{C})[-1]$ living over X

$= T_{\mathcal{C}}[-1]\mathcal{M}_{\text{CYcat}}$

Holographic Principle: The gravity thy couples to the brane gauge thy.

(Mathematically: $HC_\cdot^-(\mathcal{C}) \curvearrowright \mathrm{End}_{\mathcal{C}}(\mathcal{F})$.)

(Rmk: $HH^\cdot(\mathcal{C}) \curvearrowright \mathrm{End}_{\mathcal{C}}(\mathcal{F})$) ← (use d-CY structure to relate $HH_\cdot(\mathcal{C}) \simeq HH^\cdot(\mathcal{C})[d]$ "Every coupling is a field")

And the theories should couple uniquely in the limit $N \to \infty$.

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cf. [Costello–Li] [Ginot–Gwilliam–Hamilton–Zeinalian]: large N couplings ; L2T thm.

Thm (Loday–Quillen, Tsygan): $\mathcal{F} \in \mathcal{C}$ CY cat, $A = \mathrm{End}_{\mathcal{C}}(\mathcal{F})$. $\left(\mathfrak{gl}_N(A) = \mathrm{End}_{\mathcal{C}}(\mathcal{F}^{\oplus N})\right)$

$$\mathrm{Prim}\ \underset{\text{brane gauge Lie alg}}{C_\cdot(\mathfrak{gl}_\infty(A))} \simeq \underset{\text{closed-string fields}}{HC_\cdot(A)[1]}.$$

"At $N \to \infty$ limit, classical observables match in string field & gauge thy."

Ex: $\mathcal{C} = \mathrm{Fuk}(T^*M)$. If $\mathcal{C}$ is the usual wrapped Fukaya category, then

$\mathcal{C} = Sh_M(M) = \mathrm{Loc}(M) = \mathrm{Perf}_{C_\cdot(\Omega M)}$. $\Rightarrow HH_\cdot(\mathcal{C}) = C_\cdot(LM)$.

$HC_\cdot(\mathcal{C}) = C_\cdot^{S^1}(LM)$

Ex: $M = \mathbb{R}^n$ ⇝ $\boxed{C_\cdot^{S^1}(LM) = C_\cdot^{S^1}(pt) = k[\underset{\deg 2}{\beta}]}$

$HC_\cdot(\mathcal{C}) = C^{S^1}_\cdot(LM)$.

Ex: $M = \mathbb{R}^n \rightsquigarrow \boxed{C^{S^1}_\cdot(LM) = C^{S^1}_\cdot(pt) = k[\beta^{\deg 2}]}$.

$A := \mathrm{End}_{\mathrm{Fuk}(T^*\mathbb{R}^n)}(\mathbb{R}^n) = k$

[CLT]: $C^\cdot(\mathfrak{gl}_\infty(k)) = S(k[\beta][-1])$
$= \Lambda^\cdot(x_1, x_3, x_5, x_7, \ldots)$

$C^\cdot(\mathfrak{gl}_n(k)) = \Lambda^\cdot(x_1, \ldots, x_{2n-1})$

$C^\cdot(\mathfrak{sl}_n(k)) = \Lambda^\cdot(x_3, \ldots, x_{2n-1})$

Attention setup: (Introduced last week): Enhance gravity background by adding auxiliary branes.

$\rightsquigarrow$ background is CY cat $\mathcal{C}$ (generic on X) + brane $\mathcal{G} \in \mathcal{C}$ supported on $M \subseteq X$.

$\Rightarrow$ string field theory is controlled by $\mathfrak{g}_{grav} = \underbrace{\overbrace{HC^-_\cdot(\mathcal{C})(1-d)}^{\text{closed-string}}}_{\text{over } X} \ltimes \underbrace{\overbrace{\mathrm{End}_{\mathcal{C}}(\mathcal{G})}^{\text{open string}}}_{\text{over } M}$

Gauge theory now also has contribution from open strings:

$\mathfrak{g}_{gauge} = \underbrace{\mathrm{End}_{\mathcal{C}}(\mathcal{F}^{\oplus N})}_{\text{on } L} \ltimes \underbrace{(\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}^{\oplus N}, \mathcal{G}) \oplus \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{F}^{\oplus N}))}_{\text{on } M \cap L}$

$\mathrm{End}_{\mathcal{C}}(\mathcal{G}) - \mathrm{End}_{\mathcal{C}}(\mathcal{F})$ -bimodule.

Today's example: Skew Howe duality.

Let U & $W^\vee$ be the fund. representation of $\mathfrak{gl}_m$ and $\mathfrak{gl}_N$, resp., and consider

$$\mathfrak{gl}_m \curvearrowright \Lambda^\cdot(U \otimes W) \curvearrowleft \mathfrak{gl}_N$$

Howe dual: they are each other's commutators.

$$\rightsquigarrow \Lambda^p(U \otimes W) = \bigoplus S^\lambda U \otimes S^{\lambda^t} W$$

YDs w/ p boxes contained in $m \times N$ rectangle.

Have a map $U(\mathfrak{gl}_\ldots) \xrightarrow{\Phi_N} \mathrm{End}(\ldots)$

(I learned this from [Cautis-Kamnitzer-Morrison?]) kernel of Φ is weight space idempotents ...

$m \times N$ rectangle. (cannot ...) kernel of Φ is weight space idempotents of $U\mathfrak{gl}_m$ lying outside weight space support of this rep.

Have a map $U\mathfrak{gl}_m \xrightarrow{\Phi_N} \underbrace{\mathrm{End}_{U\mathfrak{gl}_N}(\Lambda^\bullet(\mathbb{C}^m \otimes W))}_{\text{look like alg. of observables in a } \mathfrak{gl}_N \text{ gauge thy coupled to some matter.}}$.

$= C^\bullet(\mathfrak{gl}_m)$!

$\Rightarrow$ In limit $N = \infty$, this is an isomorphism.

Setup: $\mathcal{C} = \mathrm{Fuk}(T^*\mathbb{R}^3)$, $x_1, x_2, x_3, y_1, y_2, y_3$ w/ no wrapping at ∞ for L or M. Auxiliary branes m copies of $M = \mathbb{R}^3_{\{y=0\}}$ + Stack of N branes on $L = T^*_{\{x_1=x_2=0\}}\mathbb{R}^3 = \mathbb{R}_{x_3} \times \mathbb{R}^2_{y_1, y_2}$

$\mathrm{End}(L) \simeq C^\bullet(L) \simeq k$ w/ no wrapping

$\mathcal{L}_{grav} = \begin{pmatrix} \text{Closed-string part} \\ \text{on } X \end{pmatrix} \ltimes \begin{matrix} \mathfrak{gl}_m \\ \text{on } M \end{matrix}$

$\mathcal{L} = \mathfrak{gl}_m(\Omega^\bullet_M)$ ← new open-string sector in the gravitational background is just 3d CS thy on $M = \mathbb{R}^3$.

$\mathcal{L}_{gauge} = \underbrace{\mathrm{End}_{\mathrm{Fuk}}(L^{\oplus N}) \text{ on } L}_{\mathfrak{gl}_N(\Omega^\bullet_L)} \ltimes \underbrace{\left(\mathrm{Hom}(L^{\oplus N}, M^{\oplus m}) \oplus \mathrm{Hom}(M^{\oplus m}, L^{\oplus N})\right)}_{\text{living on } L \cap M = \cancel{\mathbb{R}_{x_3}}\ \mathbb{R}^2} \otimes \Omega^\bullet_{\mathbb{R}_{x_3}}$

3d CS on L

$\left(\begin{matrix} \mathrm{Hom}(k^m, k^N) \otimes \mathrm{Hom}_{\mathrm{Fuk}}(M, L) \\ \mathrm{Hom}(k^N, k^m) \otimes \mathrm{Hom}_{\mathrm{Fuk}}(L, M) \end{matrix} \right)$

$= \left(\begin{matrix} U \otimes W \\ (U \otimes W)^\vee (-2) \end{matrix} \right)$ (-1)

$\Rightarrow \mathcal{L}_{gauge} = \mathfrak{gl}_N(\Omega^\bullet_L) \ltimes \left(\underbrace{U \otimes W \oplus (U \otimes W)^\vee(-2)}_{\boxed{T^*[-1](U \otimes W)}\,[-1]} \right) \otimes \Omega^\bullet_{L \cap M}$

3d CS ... B-model ...

3d CS coupled to 2d B-model ~~PSM~~ w/ target ~~[illegible]~~ $u\otimes W$

$$\left(T^*[-1]\left(u\otimes W\right)\right)[-1] \otimes \Omega^\bullet_{L\cap M}$$

$C^\bullet(\mathfrak{gl}_N)^{Ez!} \overset{\sim?}{\longrightarrow} PV^\bullet(\mathbb{C}^N \otimes \mathbb{C}^m)^{\mathfrak{gl}_N}$

$\Rightarrow$ Total gauge thy is 3d $\mathfrak{gl}_N$ CS along L, coupled to 1d thy along $L\cap K$.

Gauged TQM: TQM w/ target $T^*\left(\frac{u\otimes W(1)}{\mathfrak{gl}_N}\right)$ w/ the stacky $T^*(u\otimes W(1))/\mathfrak{gl}_N$.

On $L\cap K$: TQM to $T^*(u\otimes W(1))$,

Q: What is the alg. of operators of the quantization?

$T^*(u\otimes W(1))$ quantizes to Weyl-Clifford alg $Cl(T^* u\otimes W(1))$

12 Fermionic Fock space

$\mathrm{End}_k(\Lambda^\bullet u\otimes W)$.

Symplectic fermion

Quotient ~~[illegible]~~ by $\mathfrak{gl}_N$

$\rightsquigarrow \mathrm{End}_k(\Lambda^\bullet u\otimes W)^{\mathfrak{gl}_N}$

$\shortparallel$

~~[illegible]~~ $\mathrm{End}_{\mathfrak{gl}_N}(\Lambda^\bullet u\otimes W)$

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derived conifold

Ex: $\mathcal{C} = \mathrm{Fuk}(T^*S^3)$, N branes on $L = S^3$, auxiliary brane on $\Lambda = T^*_{knot} S^3$

$\rightsquigarrow$ gauge thy on the brane is

$\mathfrak{gl}_N$ 3d CS on S^3

Gopakumar-Vafa: "knot invts $\leftrightarrow$ top. strings"

$$\mathfrak{gl}_N(C^\bullet(S^3)) \ltimes \mathrm{Hom}(L,\Lambda)\otimes \mathrm{Hom}(\Lambda, L)$$

on $S^3 \otimes \Omega^\bullet_{S^3}$

Wilson line in CS thy

Naively, gauge thy is

$$HC_- = C^{S^1}_\bullet(LS^3) \ltimes \mathrm{End}_{\mathrm{Fuk}}(\Lambda)$$

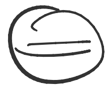
Legendrian dga ?

~~+ backreaction $\rightsquigarrow$ resolved conifold~~ ?

$\mathcal{F} = \mathbb{R}^n$

$\mathrm{End}(\mathcal{F}) = k = C^\bullet(\mathbb{R}^n)$ $\quad C^\bullet(\mathbb{R}^n) = k$

$\Rightarrow \ \mathrm{End}(F^{\oplus N}) = \mathfrak{gl}_N(\mathrm{End}(F)) = \mathfrak{gl}_N(k)$

 $S^3 \subseteq T^*S^3$. $\quad \mathrm{End}_{\mathrm{Fuk}(T^*S^3)}(S^3)^{\oplus N}$ has a canonically perturbative description, local on S^3, as $\mathfrak{gl}_N(\Omega^\bullet_{S^3})$