A users guide to holography

following Costello, Paquette, Gaiotto, Li, ...

Most famous holographic statement:

Type IIB string on $AdS_5 \times S^5$ $\cong$ $N=4$ SYM w/ $G = U(N)$ on S^4 as $N \to \infty$.

$\|$ $\|$

"Gravity" in the <u>bulk</u> $\cong$ "Gauge thy" on the boundary.

This is just <u>one</u> example. Kevin mentioned many others.

We want to formulate such a statement in terms of algebras of operators.

① Local operators. Start w/ a translation invariant theory on $\mathbb{R}^n$, w/ a fundamental field

$$\varphi \in C^\infty(\mathbb{R}^n, V). \quad V = \mathbb{C}$$

Linear local operators at $0 \in \mathbb{R}^n$,

$$\varphi \longmapsto \partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} \varphi(0).$$

$$k_j \geq 0.$$

$\Rightarrow$ Generate free commutative algebra.

- In general, the field φ must solve some equation of motion.

$\Rightarrow$ some operators are trivial.

Ex: $\varphi \in C^\infty(\mathbb{R})$ w/ EOM

$$\partial_x^2 \varphi = 0.$$

Then only non-trivial operators

$$\varphi \longmapsto \varphi(0)$$

$$\varphi \longmapsto \partial_x \varphi(0).$$

- Also, there might be gauge symmetries. Not all $\mathcal{O}$'s are invariant!

Ex : $A \in \Omega^1(\mathbb{R}^2, \mathfrak{g})$

$c \in C^\infty(\mathbb{R}^2, \mathfrak{g})$

EOM : $dA + \frac{1}{2}[A, A] = 0$.

Gauge: $\delta A = d\underline{c} + [\underline{c}, A]$.

Will impose this cohomologically, really at the cochain level.

Let:

$$\Omega^\bullet(\mathbb{R}^2, \mathfrak{g}) = \underbrace{\overset{\underline{0}}{\Omega^0 \otimes \mathfrak{g}}}_{c} \xrightarrow{d} \underbrace{\overset{\underline{1}}{\Omega^1 \otimes \mathfrak{g}}}_{A} \xrightarrow{d} \underbrace{\overset{\underline{2}}{\Omega^2 \otimes \mathfrak{g}}}_{A^+}$$

$$[\alpha \otimes X, \beta \otimes Y] = (\alpha \wedge \beta)[X, Y]_{\mathfrak{g}}$$

$\Rightarrow$ dg Lie algebra.

$MC = \{\text{flat } G\text{-bundles}\} / \text{gauge}.$

Observables on $D \subseteq \mathbb{R}^2$

$\|$ defn

$$\lim_{D \ni 0}\left[\begin{array}{cc} C^\bullet_{\mathrm{Lie}}\left(\mathcal{V}^\bullet(D)\otimes\mathfrak{g}\right) & \\ \uparrow & D \\ & \cup\!\mid \\ C^\bullet_{\mathrm{Lie}}\left(\mathcal{V}^\bullet(D')\otimes\mathfrak{g}\right) & D' \end{array}\right]$$

$$\begin{array}{c} \uparrow \\ \vdots \end{array}$$

local operators

Actually for any D

constants $\quad \underline{\mathfrak{g}} \xhookrightarrow{\sim} \mathcal{V}^\bullet(D) \otimes \mathfrak{g}$

$$X \longmapsto 1 \otimes X$$

$\Rightarrow$ local op's $\simeq$ $C^{\cdot}_{\text{Lie}}(\mathfrak{g})^{\circlearrowleft d_{CE}}$.

This was all classical. Quantum observables give rise to various versions of "non-commutative" algebras:

1) 1d (QM) the operators form a (dg / A_∞) algebra.

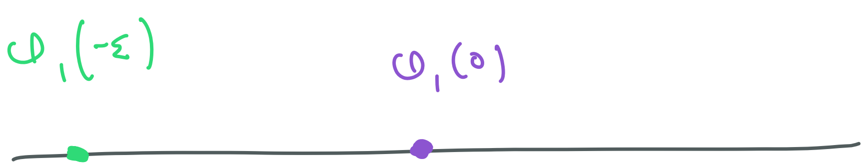
2) 2d chiral theories form a vertex algebra.

3) The local op's of a TFT

on $\mathbb{R}^n$ form a E_n-algebra.

1d QM

Assume toplogical



$$[\mathcal{O}_1, \mathcal{O}_2] = \lim_{\varepsilon \to 0^+} \mathcal{O}_1(0)\,\mathcal{O}_2(\varepsilon) - \lim_{\varepsilon \to 0^-} \mathcal{O}_1(0)\,\mathcal{O}_2(\varepsilon).$$

From the path integral this is computed by choosing a propagator and evaluating Feynman graphs.

Ex: Top^l mechanics

$$\gamma, \beta \in \Omega^{\bullet}(\mathbb{R}),$$

action $\int \beta \, d\gamma$.

EOM: $d\gamma = d\beta = 0$.

local operators:

$$q : \gamma \longmapsto \gamma(0)$$

$$p : \beta \longmapsto \beta(0).$$

The "propagator" is

$$P(x, y) = \Theta(y - x)$$

$$\in C^{\infty}(\mathbb{R} \times \mathbb{R} - \Delta).$$

Satisfies,

$$d\, P(x, y) = \delta(x - y).$$

$$\overbrace{p(0) \qquad q(\varepsilon)}^{P(0,\varepsilon)} = \Theta(\varepsilon)$$

$$\overbrace{p(0) \qquad q(-\varepsilon)}^{P(0,-\varepsilon)} = \Theta(-\varepsilon).$$

$$\Rightarrow \quad [p, q] \propto +1.$$

② Koszul duality.

A = associative algebra.

(possibly dg) / $\mathbb{C}$

$\varepsilon : A \longrightarrow \mathbb{C}$ augmentation.

map of algebras

$\mathbb{C}_\varepsilon \circlearrowleft A$, $\quad a \cdot \lambda = \varepsilon(a) \lambda$.

- A-mod is a dg category.

For $M, N \in A\text{-mod}$,

$\mathbb{R}\mathrm{Hom}_A(M, N)$. $\quad$ | $H^\bullet(-) = \mathrm{Ext}^\bullet_A(M, N)$.

= derived homomorphisms.

Take a projective, free resolution $\tilde{M}^\bullet$ of M, then

$$\mathbb{R}\mathrm{Hom}_A(M, N) \simeq \mathrm{Hom}_A(\tilde{M}^\bullet, N).$$

<u>Defn</u>: If A, ε above. Define

$$A^! \overset{\text{def}}{=} \mathbb{R}\mathrm{Hom}_A(\mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon).$$

Has dg algebra structure. Call $A^!$ the <u>Koszul dual</u> of A.

Ex: $A = \mathbb{C}[x]$, $\varepsilon(f) = f(0)$.

"Koszul" resolution for $\mathbb{C}_\varepsilon$:

$$\begin{array}{ccc} \underline{-1} & & \underline{0} \\ \xi\,\mathbb{C}[x] & \xrightarrow{x\cdot} & \mathbb{C}[x] \end{array} \circlearrowleft \mathbb{C}[x].$$

$$\| $$

$$\left(\mathbb{C}[x, \xi],\ d = x\partial_\xi\right).$$

Then,

$$\mathbb{R}\mathrm{Hom}_{\mathbb{C}[x]}(\mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon)$$

$$\simeq \mathrm{End}_{\mathbb{C}[\xi]}\left(\mathbb{C}[x, \xi]\right). \qquad [\partial_x, x] \neq 0.$$

$$\cong \left(\mathbb{C}[x, \xi, \partial_\xi],\ d = [x\partial_\xi, -]\right)$$

$|\xi| = -1$

$|\partial_\xi| = +1$

$$\uparrow \quad \simeq$$

$$\left(\mathbb{C}[\partial_\xi] \;,\; d = 0 \right).$$

$$= \quad \frac{0}{1} \qquad \frac{1}{\partial_\xi} .$$

$\varepsilon^i = \partial_{\xi_i}$

$|\varepsilon^i| = +1.$

More generally,

$$\begin{array}{ccc} \mathbb{C}[x_1, \dots, x_n]^! & \simeq & \mathbb{C}[\varepsilon^1, \dots, \varepsilon^n] \\ \| & & \| \\ \mathrm{Sym}(V)^! & \simeq & \Lambda(V^*) \\ & & \| \quad \mathbb{Z}/2 \\ & & \mathrm{Sym}(V^*[-1]) \end{array}$$

graded symmetric algebra.

Ex: $\mathfrak{g} = \mathrm{Lie}(G)$.

$C^\cdot(\mathfrak{g}) \cong$ left inv't differential forms on G.

$\circlearrowleft$ $\circlearrowleft$

d_{CE} d_{dR}.

For any $U \subseteq G$ have

$$\Omega^\cdot(U) \cong C^\infty(U) \otimes \Lambda^\cdot(\mathfrak{g}^\vee)$$

since $TG \cong G \times \mathfrak{g}$. If

$\hat{G}$ denotes neighborhood of $1 \in G$, then

$$\mathbb{C} \xhookrightarrow{\simeq} \Omega^\cdot(\hat{G})$$ by Poincaré lemma.

wedge product.

$$\Omega^\cdot(G)^G \cong C^\cdot(\mathfrak{g})$$

$\Rightarrow$ $\Omega^\cdot(\hat{G})$ is free resolution for trivial module $\mathbb{C}$. So,

$$C^\cdot(\mathfrak{g})^! \simeq \operatorname{End}_{\Omega^\cdot(G)^G}\left(\Omega^\cdot(\hat{G})\right).$$

$\wr$

Operators which commute w/ wedge product by left inv't differential.

$X \in \mathfrak{g}$, $\mathcal{L}_X$ = Lie derivative generated by inf. left translation.

Also, products of these.

$$[\mathcal{L}_X, \mathcal{L}_Y]_{vf} = \mathcal{L}_{[X,Y]_{\mathfrak{g}}}$$

$$\Rightarrow \quad \mathcal{U}\mathfrak{g} \hookrightarrow C^{\cdot}(\mathfrak{g})^{!}$$

$$\mathcal{U}\mathfrak{g} = \mathrm{Tens}(\mathfrak{g}) \big/ x\cdot y - y\cdot x - [x,y].$$

Can show this is $\simeq$.

• Simplest toy model of "holography".

① "Gravity" theory on $\mathbb{R} \times \mathbb{R}^n$.

Local operators $= \mathcal{A}$

② Place a "stack of branes" on

$$\mathbb{R} \times \{0\}$$

This is a line defect in bulk thy

Local operators along defect $= \mathcal{B}(N, \dots)$

There is a canonical map of alg's

$$\phi_{N,\dots} : \mathcal{A}^! \longrightarrow \mathcal{B}(N, \dots).$$

Statement of holography is that $\phi_{N,\dots}$ becomes equivalence in some limit

To connect w/ the more standard picture ...

Remove locus of brane.

$$\mathbb{R} \times \mathbb{R}^n \setminus \mathbb{R} \times \{0\} \simeq \mathbb{R} \times \mathbb{R}_{>0} \times S^{n-1}$$

$$\downarrow$$

$$\mathbb{R}_t \times \mathbb{R}_{r>0}$$

$\rightsquigarrow$ Effective 2d thy. BC at

$$\{r=0\} \rightsquigarrow \mathcal{A} = \mathcal{A}_{r=0}$$

At $r=\infty$ we assume that "gravity" admits a BC s.t. $\mathcal{A}_{r=\infty} \overset{!}{\simeq} \mathcal{A}_{r=0}$.

So, we get

$$\mathcal{A}_{r=\infty} \longrightarrow \mathcal{B}(N, \dots)$$

Why equivalence as $N \to \infty$?

Original argument is based on string thy, which I won't touch on. The key idea is that the gravity side operators are Koszul dual to some "universal" brane thy.

3 Couplings.

Two independent QM systems $\mathcal{A}, \mathcal{B}$

$$\boxed{\mathcal{A} \otimes \mathcal{B}} \quad , \quad \mathcal{A}, \mathcal{B} \text{ dg alg's of local op's.}$$

Defn: An algebraic coupling is

a deformation of differential

$$Q_A + Q_B + \tilde{Q}$$

when $\tilde{Q} : \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{A} \otimes \mathcal{B}$ deg $+1$.

- $\tilde{Q}$ derivation

satisfies $\cdot \; (Q_A + Q_B + \tilde{Q})^2 = 0.$

We will assume further that

$$\boxed{\tilde{Q} = [\alpha, \underbrace{(-)}_{A \otimes B}].}$$

where $\alpha \in A \otimes B$ degree $+1$.

Claim : α satisfies

$$\left[Q_A \alpha + Q_B \alpha + \alpha \cdot \alpha = 0. \right]$$

$\Longleftrightarrow$ α is a Maurer - Cartan elmt in

dg Lie algebra $\underline{A \otimes B}$.

$\ker(\varepsilon) \subseteq A$ augmentation ideal

$\alpha \in \ker(\varepsilon) \otimes B$

Theorem : Sps $\varepsilon : \mathcal{A} \to \mathbb{C}$ is an augmentation. Then there is 1-1 correspondence

"Algebraic couplings"

- MC elements $\alpha \in \mathcal{A} \otimes \mathcal{B}$.
  s.t. $(\varepsilon \otimes \mathbb{1})(\alpha) = 0$.
- Algebra homomorphisms

$$\phi_\alpha : \mathcal{A}^! \longrightarrow \mathcal{B}.$$

$\xrightarrow{\varepsilon} \mathrm{Sym}^0 \cong \mathbb{C}$

Ex : $\mathcal{A} = C^\bullet(\mathfrak{g})$. Sps $\mathcal{B}$ is an ordinary (non dg) algebra.

$\Rightarrow$

$$\alpha \in \underline{\mathfrak{g}^* \otimes \mathcal{B}} \subseteq C^\bullet(\mathfrak{g}) \otimes \mathcal{B}.$$

$\mathfrak{g}^* \cong C^1(\mathfrak{g})$.

$$d_{CE}\,\alpha + \alpha \cdot \alpha = 0$$

$\Longleftrightarrow$ Equivalent to $\tilde{\phi}_\alpha : \mathfrak{g} \longrightarrow \mathcal{B}$ s.t.

$$\tilde{\phi}_\alpha\left([x,y]_{\mathfrak{g}}\right) = \underbrace{\tilde{\phi}_\alpha(x) \cdot_{\mathcal{B}} \tilde{\phi}_\alpha(y) - \tilde{\phi}_\alpha(y) \cdot_{\mathcal{B}} \tilde{\phi}_\alpha(x)}. \qquad \forall\, x, y$$

In other words $\tilde{\phi}_\alpha$ is a Lie map

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\tilde{\phi}_\alpha"] \arrow[d, hook] & \mathcal{B} \\
U\mathfrak{g} \arrow[ur, dashed, "\exists!\, \phi_\alpha"'] &
\end{tikzcd}$$

$\Rightarrow$ $U\mathfrak{g}$ is the algebra of operators on the universal line defect.

$\mathcal{B}$ = QM on line

$\mathcal{A}$ = loc. op's "gravity"

There is $\alpha_{univ} \in \mathcal{A} \otimes \mathcal{A}^!$.

In turn $\Longleftrightarrow$ $\mathbb{1} : \mathcal{A}^! \longrightarrow \mathcal{A}^!$

Given another coupling, MC element

$$\alpha \in A \otimes B$$

have

$$\alpha = \Phi_\alpha (\alpha_{univ}) .$$

$A^!$ is the alg of operators on the universal line defect in the gravitational thy A.

Physically, the choice of an augmentation is a vacuum in the gravitational thy.

$\varepsilon(\alpha) = 0 \iff$ line defect preserves this vacuum.

• Relationship to "physical" couplings.

$\mathcal{O} \in \mathcal{A}$ local operator at $0 \in \mathbb{R}^n$.

$\| $

$\mathcal{O}(0)$ $\qquad x \in \mathbb{R}^n$.

Consider $\tau_x \mathcal{O} \overset{\text{def}}{=} \mathcal{O}(x)$. Define

$$\mathcal{O}(x) \in C^\infty(\mathbb{R}^n, \mathcal{A}).$$

Sps we are in a TFT. Then

$$\frac{\partial}{\partial x_i} \text{ acts htpy trivially.} \quad i = 1, \ldots, n$$

$(\mathcal{A}, Q_\mathcal{A})$ dg algebra.

η^i deg -1.

$\Rightarrow \exists\ \eta^i$ s.t.

$$[Q_\mathcal{A}, \eta^i] = \frac{\partial}{\partial x_i}$$

$$\omega^{(1)}(x) \overset{\text{def}}{=} \sum_i \left(\eta^i \omega \right)(x)\, dx_i$$

$$\in \Omega^1(\mathbb{R}^n, A)$$

satisfies "descent"

$[Q_A, \eta^i] = \partial_{x_i}$.

$$d_{dR}\, \omega^{(0)}(x) = Q_A\, \omega^{(1)}(x).$$

Can go all the way up ...

$$\omega^{(n)}(x) = \left(\eta^i \omega^{(n-1)} \right)(x)\, dx_i$$

Defines Lagrangian

$$\int_{\mathbb{R}^n} \omega^{(n)}(x)$$

Automatic $Q_{\mathcal{A}}$ - closed :

$$Q_{\mathcal{A}} \int \omega^{(n)}(x) = \int d_{dR}(-)$$

$$= 0 .$$

For line defects we will just care about $n = 1$.

$$\alpha \in \mathcal{A} \otimes \mathcal{B} \quad \text{degree } +1 .$$

$\Rightarrow$

$$\alpha^{(1)} \in \Omega^{1}(\mathbb{R}, \mathcal{A})$$

$\Rightarrow$

$$\int_{\mathbb{R}} \alpha^{(1)} \quad \text{Lagrangian density.}$$

Automatically is BRST invariant to all orders in perturbation thy.

$\underline{Ex}$: $A = C^\bullet(\mathfrak{gl}_N) = \mathbb{C}[c_a]$.

deg +1

d_{CE}

$B = \mathrm{Weyl}_N$
$\qquad\;\; \| $
$\qquad \mathbb{C}[p^i, q_j]$, $[p^i, q_j] = 1$.

ordinary alg

Defining $\mathfrak{gl}_N$ action defines MC:

$$\alpha = \rho_i^{aj} c_a p^i q_j \in A \otimes B.$$

$\mathfrak{gl}_N^* \otimes \mathrm{Weyl}_N$.

$C^\bullet(\mathfrak{gl}_N)$ is the algebra of local operators of a 1-dim^l gauge field : $\Omega^\cdot(\mathbb{R}) \otimes \mathfrak{gl}_N$.

$$\begin{tikzcd} \underline{0} & & \underline{1} \\ \Omega^0(\mathbb{R}) \otimes \mathfrak{gl}_N \arrow[rr, "d"] & & \Omega^1(\mathbb{R}) \otimes \mathfrak{gl}_N \arrow[ll, bend left, "\eta"] \\ c \arrow[u, phantom, sloped, "\in"] & & A \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

$$\underset{\eta}{\overset{i_{\partial/\partial t}}{\shortparallel}}(f\,dt) = f \qquad [d,\eta] = \frac{\partial}{\partial t}$$

Then

$$\alpha^{(1)} = \rho^{ai}_{j} A_a p^j q_i .$$

$\Rightarrow$ Lag coupling

$$\int_{\mathbb{R}} \rho^{ai}_{j} A_a p^j q_i = \int_{\mathbb{R}} (p, A \cdot q) .$$

(4) Baby holography

Consider $\mathfrak{gl}_N$ CS thy on

$$\mathbb{R} \times \mathbb{R}^2 .$$

And some "branes" along $\mathbb{R} \times \{0\}$.

$$\mathbb{R} \times \mathbb{R}^2 - \mathbb{R} \times \{0\}$$

$$\simeq \mathbb{R} \times \underbrace{\mathbb{R}_{>0} \times S^1}_{\text{cylinder}}$$

$$\downarrow$$

Poisson σ-model $\rightsquigarrow$ $\mathbb{R} \times \mathbb{R}_{>0}$

or

"BF" thry. $\xleftarrow{\int_{S^1}}$ CS

$$\int B F_A$$

$$A \in \Omega^{\cdot}(\mathbb{R} \times \mathbb{R}_{>0}) \otimes gl_N[1]$$

$$B \in \Omega^{\cdot}(\mathbb{R} \times \mathbb{R}_{>0}) \otimes gl_N.$$

- BC at $r = 0$:

$$\{B = 0\} \qquad \mathbb{R} \times \{0\}$$

$\Rightarrow$ local operators of just the gauge field $\simeq C^{\cdot}(g)$.

- BC at $r = \infty$:

$$\{A = 0\}$$

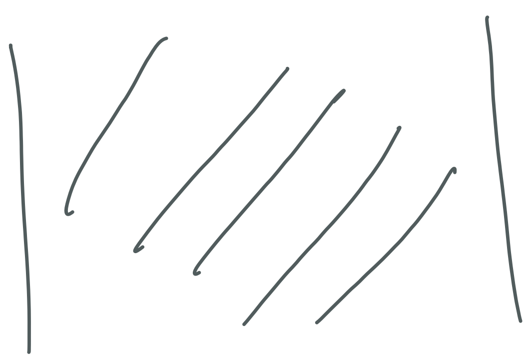
Operators of just the B-fields.

$\simeq \mathcal{U}g$.

$$\begin{tikzcd} C^\cdot(\mathfrak{g})^! \arrow[r, "\simeq"] \arrow[d, equal] & U\mathfrak{g} \arrow[d, equal] \\ A^! \arrow[r, "\simeq"] & \mathcal{B}_\infty \end{tikzcd}$$

$A = C^\cdot(\mathfrak{g})$ local op's in CS = "gravity"

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$$



$N = 0$ $N = \infty$

Neumann   Dirichlet