Back reactions

"Gravity" thy on $\mathbb{R}^n$ w/ brane

along

$$\mathbb{R}^k \subseteq \mathbb{R}^n .$$

A = alg of op's for gravity.

B_N = alg of op's for brane.

Coupling : $A^! \longrightarrow B_N$.

in general gets deformed

$$\tilde{A}^! \longrightarrow B_N .$$

Koszul dual of gravity in a "modified geometry".

① Brane charges

From a worldsheet perspective

$$\Sigma \xrightarrow{\phi} X$$

branes come from boundary conditions

$$\phi\Big|_{\partial\Sigma} \in L \subset X .$$

The boundary conditions must be compatible with supersymmetry, gauge symmetries, etc...

$\Rightarrow$ Only certain submanifolds L are consistent.

Ex: • Top^l A-model

X = symplectic mfld

$L \subset X$ Lagrangian.

• Top^l B-model

X = cplx mfld.

$L \subset X$ cplx submfld

(better: coherent sheaf).

We want to think about branes as **defects** in the target space they on X.

• **Sources**: In general, there will be fields in the gravitational spacetime^{X} thy which "source" a brane.

In physical string thy, there are fields, called potentials, in the thy

$$C^{(p)} \in \Omega^{p}(X).$$

Which source a p-dim'l brane $L \subset X$ via

$p=1 \rightsquigarrow$ Wilson line

$$\int_{L \subset X} C^{(p)} = \int_{X} \delta_{L \subset X} \wedge C^{(p)}.$$

Often EOM only involve the field strength the $dC^{(p)} \in \Omega^{p+1}(X)$.

If X^d is Riemannian, then

$$* : \Omega^k(X) \longrightarrow \Omega^{d-k}(X).$$

The electric-magnetic dual of $C^{(p)}$ is

$$\tilde{C}^{(d-p-2)} \in \Omega^{d-p-2}(X).$$

s.t.

$$d\tilde{C} = * dC$$

Say that

- $C^{(p)}$ "electrically sourced" for $\mathcal{L}^p \subseteq X$. $\int_{\mathcal{L}} C^{(p)}$
- $C^{(p)}$ "magnetically source" for $\mathcal{L}^{d-p-2} \subseteq X$.

$$\int_{\mathcal{L}^{d-p-2}} \tilde{C}^{(d-p-2)}.$$

Ex: In IIA string the following field strengths appear dc RR fields.

$$F^{(0)}, \ F^{(2)}, \ldots, \ F^{(10)} .$$

In IIB string they,

$$F^{(1)}, \ F^{(3)}, \ldots, \ F^{(9)} .$$

Called Ramond-Ramond field strengths.

Ex: In IIB there is $\{F^{(1)}, F^{(3)}, F^{(5)}_{+}\}$.

$$F^{(10-2k-1)} = * F^{(2k+1)}$$

$A^{(2k)}$ is RR form $dA^{(2k)} = F^{(2k+1)}$

couples to $D(2k-1)$ branes magnetically

$$\int_{\mathbb{R}^{2k}} A^{(2k)} = \int_{\mathbb{R}^{2k}} d^{-1} F^{(2k+1)} .$$

↖ fund. field.

$A^{(2k)}$ is YM type field, Kinetic part is

$$\int dA^{(2k)} \wedge \underbrace{* \, \overbrace{dA^{(2k)}}^{F^{(2k+1)}}}_{F^{(10-2k-1)}}$$

In presence of magnetic coupling,

$$dF^{(10-2k-1)} = \delta_{\mathbb{R}^{2k} \subseteq \mathbb{R}^{10}}.$$

$$\int_{\mathbb{R}^{2k}} A^{2k} = \int_{\mathbb{R}^n} \delta_{\mathbb{R}^{2k}} \wedge A^{2k}.$$

<u>Upshot</u>: Presence of brane modifies the EOM for the purely gravitational theory.

Rough idea: Forget about thy **on** the brane. The presence of the brane means that certain fields in gravity acquire charges, meaning EOM are modified.

We should be computing local operators of this modified gravitational thy.

$$\mathcal{A}(X\text{-brane}) \rightsquigarrow \tilde{\mathcal{A}}_N(X\text{-brane}).$$

The charge will depend on the # of branes (and possibly other data ...)

$$\int A^{(2k)} \wedge \left(N\, \delta_{\mathbb{R}^{2k}}\right)$$

② Back reactions as deformations.

Perturbatively, a classical field thy is described by an L_∞ algebra

$$\left(\mathcal{L} \; ; \; l_1 \, , \, l_2 \, , \ldots \right)$$

$$l_k : \mathcal{L}^{\times k} \longrightarrow \mathcal{L}[2-k] .$$

- $l_1^2 = 0$,

- $l_2 \circ l_1 = l_1 \circ l_2 \quad \ldots$

At level of action fnls

pairing

$$\int_X \omega(\alpha, l_1 \alpha) + \omega(\alpha, l_2(\alpha \alpha)) + \ldots$$

Perturbative field thy $\rightsquigarrow$ L_∞ algebras
in BV fmls w/ "cyclic str."

- S : (fields) $\to \mathbb{C}$
- $\{-,-\}$ BV bracket

- $\mathcal{O}(\text{fields}) \times \mathcal{O}(\text{fields}) \to \mathcal{O}(\text{fields})[1]$

$$\{S,S\} = 0$$

"classical master Eqn"

$$\{-,-\} \text{ "=" } \omega^{-1}$$

$$S = \int \omega(\phi, l_1 \phi) + \int \omega(\phi, l_2 \phi) + \dots$$

$$\{S,S\} = 0 \iff L_\infty \text{ relns.}$$

MC eqn for $\mathcal{L}$ $\iff$ EOM

$$D(\alpha) = 0.$$

↖ non-lin PDE.

Sps α sourced by a brane $\mathcal{L} \subset X$,

then there is a term in the

Lagrangian

$$\int_{\mathcal{L}\subset X} \alpha = \int_X \alpha \wedge \delta_{\mathcal{L}\subset X}.$$

EOM get's modified

$$D(\alpha) = \delta_{\mathcal{L}\subset X}. \qquad (*)$$

One way to think about this

is as a "MC eqn" for a

curved L_∞ algebra

Note that the curving is localized to the brane.

A soln to $(*)$ is called a "back reaction" $\alpha = \alpha_{BR}$.

α_{BR} will have singularities along the brane. $\rightsquigarrow$ Defines the thy away from locus of brane.

$X \setminus \mathcal{L}$ Background where α takes nonzero value α_{BR}.

This discussion ignores the actual theory along the brane.

In practice we have

$$\overset{\alpha}{\mathcal{E}_{grav}} \oplus \overset{A}{\mathcal{E}_{brane}}$$

$$\int_X \mathcal{L}_{grav}(\alpha) + \int_{L \subset X} \alpha$$

$$\int_{L \subset X} \mathcal{L}_{couple}(\alpha, A) + \int_{L \subset X} \mathcal{L}_{brane}(A).$$

In this sense, the source term is like a "zeroth" order coupling to the thy on the brane.

Two points of view :

1) Thy on $X \times \mathcal{L}$ is deformed.

Alg of operators at ∞ in this new background

$$\tilde{\mathcal{A}}_\infty \xrightarrow{\simeq} \lim_{N \to \infty} B_N$$

?$\|$

$\tilde{\mathcal{A}}^!$ for some $\tilde{\mathcal{A}}$ along the brane ??

2) In the presence of brane, thy is anomalous, but this anomaly can be trivialized.

$$\rightsquigarrow \mathcal{A}^! \Big|_{\hbar = f(N)} = \mathcal{A}_\infty .$$

• In the magnetically coupled case, things are trickier.

Sps α is a $(p+1)$-form. Then a magnetic source for p-dim^l will look like

$$\int_{\mathcal{L}^p \subset X} d^{-1}\alpha \quad .$$

Two points of view:

1) Thy on $X \setminus \mathcal{L}$ is still defined

$\delta(\partial^{-1}\alpha)$: $\quad D(\alpha, d\alpha, \dots) = \delta_{\mathcal{L}\subset X}.$

$\rightsquigarrow \alpha_{BR}$

$\rightsquigarrow \tilde{A}_x$ alg of op's at

$x \in X \setminus \mathcal{L}$ in this background.

2) Theory is anomalous. The anomaly defines some central extension of the gravitational fields along the brane.

$$\mathbb{C} \longrightarrow \widehat{\text{fields}} \longrightarrow \text{fields}$$

anomaly gives rise to central ext.!

$$A = C^{\cdot}(g) \longleftarrow$$

$$\hat{A} = C^{\cdot}(\hat{g}) .$$

Have

$$\tilde{A}_{\partial} = \hat{A} !$$

③ Examples Let's use the following toy "gravitational" model.

$$\mathbb{R} \times \mathbb{C}^2$$

top'l holomorphic

$$\alpha \in \Omega^{\cdot}(\mathbb{R}) \otimes \Omega^{0,\cdot}(\mathbb{C}^2) \otimes \mathfrak{gl}_n \quad [1]$$

$$S(\alpha) = \frac{1}{2} \int d^2 z \wedge \left(\alpha \, d\alpha + \frac{1}{3} \alpha \{ \alpha, \alpha \} \right)$$

where

$\{\cdot , \cdot\}$ P.B. on $\mathbb{C}^2$.

This theory is sick past one-loop.

Who cares...

First type of brane:

"M2 branes"

- $\mathbb{R} \times \{0\} \subseteq \mathbb{R} \times \mathbb{C}^2$

Source term $N \int_{\mathbb{R} \times \{0\}} \alpha = N \int \alpha \wedge \delta_{\mathbb{R}}$

(α: 1-form, $\delta_{\mathbb{R}}$: 4-form.)

Leads to curved MC eqn / EOM

$$(\not{d} + \bar{\partial}) \alpha + \frac{1}{2} \{\alpha, \alpha\} = N (d^2 z)^{-1} \delta_{\mathbb{R} \times 0}$$

(2-forms; $\{\alpha, \alpha\} \to 0$; only $d\bar{z}$)

Soln $\bar{\partial} \alpha = \delta_{\mathbb{R} \times 0}$, $\mathbb{R} \times \mathbb{C}^2 - \mathbb{R} = \mathbb{R} \times (\mathbb{C}^2 \setminus 0)$.

$$\alpha_{BR} = N \frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{|z|^4}$$

"Bochner-Martinelli" $\oint_{S^3}$

X hol symplectic:

$$\mathbb{C} \longrightarrow \Omega^{0,\bullet}(X) \xrightarrow{\partial} \Omega^{0,\bullet}(X, T)$$

($\alpha \in \Omega^{0,\bullet}(X)$; $\Omega^{hol} \longrightarrow \mathrm{Vect}$)

$\rightsquigarrow \quad \partial \alpha_{BR} \in \Omega^{0,1}(\mathbb{C}^2 \setminus 0, T)$

Beltrami

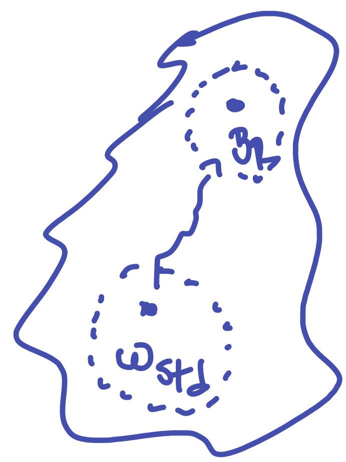
deforms cplx str. on $\mathbb{C}^2 \setminus 0$.

This

$$\mathbb{R} \times \widetilde{(\mathbb{C}^2 \setminus 0)}_N$$

is a twisted version of

$$AdS_2 \times S^3.$$



Compactify along $S^3 \subset \mathbb{C}^2 \setminus 0$ to get thy on $\mathbb{R} \times \mathbb{R}_{>0} \ni (t, |z|)$.

This is the PSM for some huge Poisson mfld, $\mathrm{Diff}(\mathbb{C})^\vee$, w/ an extra term proportional to α_{BR}. Effect:

At $|z| = \infty$: $1 \in \mathrm{Diff}(\mathbb{C}) = N$

- There is a magnetic coupling ($\ddot{A}_\alpha \big|_{1=N}$) to a brane along

$$0 \times \mathbb{C}_{z_1} \times 0 \subset \mathbb{R} \times \mathbb{C}^2 .$$

$$\alpha \in \Omega^0(\mathbb{R}) \otimes \Omega^{0,1}(\mathbb{C}^2)$$

$$\cong \quad d^2 z$$

$$\partial \quad \Omega^0(\mathbb{R}) \otimes \Omega^{2,1}(\mathbb{C}^2)$$

$$\uparrow \quad 1 \otimes \partial$$

$$\partial^{-1}\alpha \in \Omega^0(\mathbb{R}) \otimes \Omega^{1,1}(\mathbb{C}^2) .$$

Source term $\displaystyle\int_{0 \times \mathbb{C} \times 0} \partial^{-1}(d^2 z \wedge \alpha)$.

EOM in presence of brane:

$$(d + \bar{\partial})\, \partial\alpha + \frac{1}{2}\partial\{\alpha, \alpha\} = \underset{\text{3-form}}{\delta_{0 \times \mathbb{C} \times 0}} .$$

$$\int \alpha (d + \bar\partial) \underline{\alpha} + \cdots$$

$$= \int \alpha (d + \bar\partial) \underline{\partial \partial^{-1}} \alpha + \cdots$$

A soln is

$$\alpha_{BR} = \frac{\bar z_2 \, dt - \bar z_1 \, d\bar z}{(t^2 + |z_2|^2)^{3/2}}$$

This is like a deformation

of

$$\mathbb{R} \times \mathbb{C}^2 \setminus \mathbb{C}_{z_1}$$

$\mathbb{R} \times S^2$

$$\simeq \underbrace{(\mathbb{R} \times \mathbb{C} \setminus 0)} \times \mathbb{C}_{z_1}$$

as a THF manifold.

The term $\int_{\mathbb{C}_{z_1}} \partial^{-1}\alpha$ gives rise to an anomaly.

$$\{S + \int \partial^{-1}\alpha, S + \int \partial^{-1}\alpha\}_{BV}$$

$$\| \qquad \cdot\ c \in \Omega^0_{z_1}$$

$$\int_{\mathbb{C}_{z_1}} c\, \partial_{z_1} \alpha \qquad \cdot\ \alpha \in \Omega^0 \otimes \Omega^{0,1}_{z_1}$$

$$C^\cdot_{loc}(\mathcal{L}_{grav}|_{brane}).$$

$$A = C^\cdot(\Omega^\cdot(\mathbb{R}) \otimes \Omega^{0,\cdot}(\mathbb{C}^2))$$

$$\simeq C^\cdot(\underline{\mathbb{C}[z_1, z_2]}) \quad \text{P.B.}$$

As a vertex algebra along z_1 plane.

At level of KD this is like a central charge.

Consider

$$\int_{\mathbb{C}^\times \subset \mathbb{C}_{z_1}} \mathcal{A} \simeq C^\cdot(\mathbb{C}[z_1^\pm, z_2])$$

This is dg algebra. As such:

$$\left(\int_{\mathbb{C}^\times} \mathcal{A}\right)^! \simeq U(\mathbb{C}[\hbar, z_1^{-1}, z_2])$$

Have central ext of

$$\mathbb{C} \longrightarrow \mathcal{H} \longrightarrow \mathbb{C}[z_1^\pm, z_2]$$

$$(f, g) \longmapsto \oint_{z_1} f \partial g \Big|_{z_2 = 0}.$$

First term in central ext. for

$$W_{1+\infty}$$

$$\mathbb{R}\times 0 \subset \mathbb{R}\times\mathbb{C}^2$$

$$\rightsquigarrow \underbrace{\mathbb{R}\times\mathbb{C}^2 \setminus \mathbb{R}\times 0} \cong \mathbb{R}\times \widetilde{(\mathbb{C}^2\setminus 0)}_N$$

$$\cong \mathbb{R}\times \widetilde{(S^3\times\mathbb{R})}_N$$

$$0\times\mathbb{C}\times 0 \subset \mathbb{R}\times\mathbb{C}^2$$

$$\rightsquigarrow \underbrace{\mathbb{R}\times\mathbb{C}^2\setminus\mathbb{C}} \cong \widetilde{(\mathbb{R}\times\mathbb{C}\setminus 0)}_N\times\mathbb{C}$$

$$\cong \widetilde{(S^2\times\mathbb{R})}_N\times\mathbb{C}$$

"THF"

$$(\mathbb{R}\times\mathbb{C})\setminus 0$$

<u>Summary</u> : $\mathcal{A}$ = alg of op's in gravity

$\mathcal{B}_N$ = alg of op's on brane.

Sps $\mathcal{A} = C^{\cdot}(g_{grav})$. Naively,

the coupling

$$\mathcal{U}\, g_{grav} \longrightarrow \mathcal{B}_N$$

is modified by the back reaction

<u>Electric</u> : $\mathcal{A}^! = \mathcal{U}\, g_{grav} \rightsquigarrow$

$$\tilde{\mathcal{A}}^!_N = \mathcal{U}\, g_{grav} / \text{central elmt} = f(N).$$

<u>Magnetic</u> : $\mathcal{A}^! = \mathcal{U}\, g_{grav} \rightsquigarrow$

$$\tilde{\mathcal{A}}^!_N = \mathcal{U}_{\hbar}\, g_{grav} = \mathcal{U}(\hat{g}_{grav}).$$

Holography is the statement that

$$\tilde{A}^{!}_{N} \xrightarrow{\sim} B_N \quad \text{As } N \to \infty .$$

④ Computing $\tilde{A}^!$.

I now want to give a systematic approach to computing $\tilde{A}^!$.

- Ignore backreaction. Let's also assume that $A = C^\cdot(g_{grav})$.

Then, there is a canonical coupling

$$\mathbb{1} \in A \otimes A^!$$

Koszul dual along the brane.

If $\mathcal{O}$ is operator in gravity then, write $T_{\mathcal{O}}$ for corresponding element in $A^!$. Coupling is $\int_{brane} \tilde{\mathcal{O}} T_{\mathcal{O}}$.

Ex: $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{C}^2$.

Observables for theory on $\mathbb{R} \times \mathbb{C}^2$

$$\mathcal{O}[k,l] \in C^{\cdot}_{\text{linear}}(\mathfrak{g}[z_1, z_2])$$

$$c \longmapsto \partial_{z_1}^{k} \partial_{z_2}^{l} c(0,0,0).$$

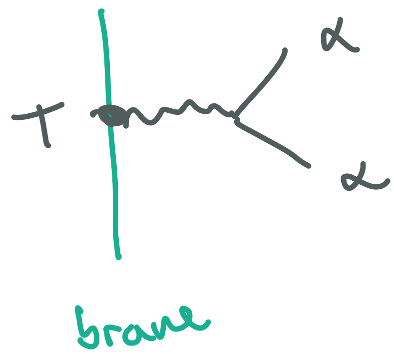
c = ghost

$$T[k,l] = z_1^k z_2^l \in U_{\text{liner}}(\mathfrak{g}[z_1, z_2]).$$

Coupling

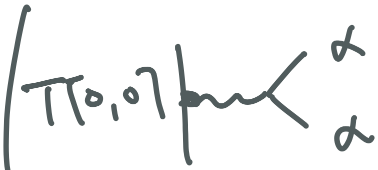
$$\int_{\mathbb{R} \times 0} \partial_{z_1}^{k} \partial_{z_2}^{l} A(t)\, T[k,l].$$

Gauge anomalies give rise to relations in $\mathcal{A}$! . Typical Feynman diagram:



Already, this diagram has an anomaly.

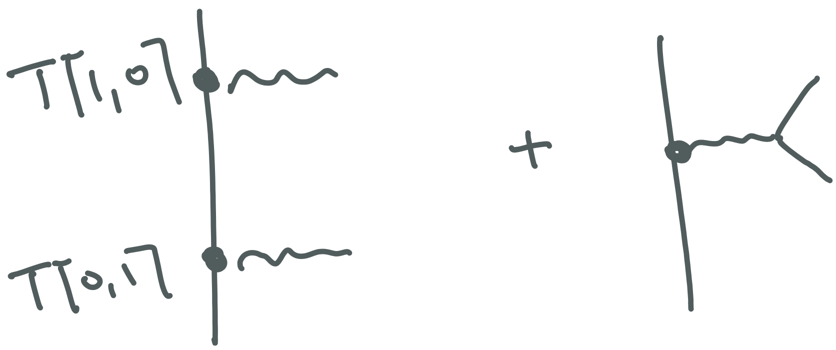
<u>Ex</u>: On $\mathbb{R} \times \mathbb{C}^2$

$\delta\Big($  $\Big) =$

$$\int_{\mathbb{R}_t} T[0,0]\, \partial_{z_1} c \; \partial_{z_2} \alpha \, (t) .$$

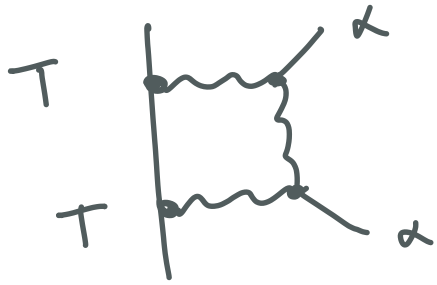
To cancel this anomaly must introduce a $T \cdot T$ OPE:

$$[T[1,0], T[0,1]] = T[0,0].$$



Is anomaly free.

These are very interesting quantum corrections.



$\rightsquigarrow [T,T] \sim T^2$

Costello, Gaiotto-Oh, Oh-Zhou

Back reaction: for $\mathbb{R} \subset \mathbb{R} \times \mathbb{C}^2$.

Have coupling

$$\int_{\mathbb{R}_t} T[0,0]\, \alpha(t).$$

In presence of the back reaction

② Interlude: couplings in BCOV

Our favorite "gravity" thy is usually built from

BCOV thy = closed string field thy of top^l B-model.

On a CY X, fields are

$$\bigoplus_{i+j \leq n} u^j PV^{i,\bullet}(X)[2] \subseteq HC^\bullet(X)[2]$$

$\circlearrowright$ $\bar{\partial} + u\partial$

$|u| = 2$

- $\dim_{\mathbb{C}} X = 3$ original defn of BCOV.

- Costello-Li: extend defn to any CY.

When $\dim_{\mathbb{C}} X = 5$,

BCOV on $\mathbb{C}^5$ $\simeq$ holomorphic twist of Type IIB SUGRA

Hol CS on $\mathbb{C}^5$ $\simeq$ holomorphic twist of SYM.

|| worldvolume thy on a D9 brane

$\downarrow$

$$\frac{1}{2}\int \Omega \wedge \left(A \bar{\partial} A + \frac{1}{3} A [A, A] \right)$$

$$A \in \Omega^{0,\bullet}(\mathbb{C}^5) \otimes \mathfrak{g}[1].$$

"BCOV is the universal thing which couples to holomorphic CS".

A coupling is

$$J \in \mathcal{O}_{loc}(\underset{\mu}{\mathcal{E}} \oplus \underset{A}{\mathcal{E}_{hCS}})$$
$$\|$$
$$\int F(\mu, A)$$

which is compatible w/ gauge symmetry in $\mathcal{E}$ and $\mathcal{E}_{hCS}$.

$\rightsquigarrow$ Satisfies the BV CME

$$d_{CE,\mathcal{E}} J \quad + \quad + \{ S_{hcs} , J \}$$

$$+ \frac{1}{2} \{ J , J \} = 0$$

Equivalently,

$$J : \mathcal{E}[-1] \overset{L_\infty\text{ map}}{\rightsquigarrow} \mathcal{O}_{loc}(\mathcal{E}_{hcs})[-1]$$

Ex: On CY3 X

$$\mathcal{E} = \mathcal{E}_{bcov}$$

-2	-1	0	1	2

$PV^{0,\cdot}$ (η)

$$\underset{\mu}{PV^{1,\cdot}} \xrightarrow{\partial} \underset{\nu}{u\, PV^{0,\cdot}}$$

$$\underset{\Pi}{PV^{2,\cdot}} \longrightarrow \underset{u\,\Pi^{(1)}}{u\, PV^{1,\cdot}} \longrightarrow \underset{u^2\,\Pi^{(2)}}{u^2\, PV^{0,\cdot}} .$$

- η couples via

$$\int_X \eta \, \mathrm{Tr}(A) \wedge \Omega$$

- μ couples via

$$\frac{1}{2} \int_X [\mu \, \nu \, \mathrm{Tr}(A \partial A)] \wedge \Omega .$$

* Not quite an allowed coupling, only if $\partial \mu = 0$. On the other hand can add

$$\frac{1}{2} \int \nu \, \mathrm{Tr}(A^3) .$$

- Π couples

$$\frac{1}{6} \int \Pi \, \mathrm{Tr}(A \partial A \partial A)$$

Again only consistent if $\partial \Pi = 0$.

Need to add

$$\int \Pi^{(1)} \operatorname{Tr}(A^3 \partial A)$$

$$+ \int \Pi^{(2)} \operatorname{Tr}(A^5).$$

In the above formulas, have been implicitly working w/ a matrix Lie algebra $\mathfrak{g}$. Now $\mathfrak{g} = \mathfrak{gl}_N$.

- LQT. A = dg algebra

$$\operatorname{Sym}[\operatorname{Cyc}^\bullet(A)[-1]] \xrightarrow{\simeq} C^\cdot(\mathfrak{gl}_\infty(A))$$

$$\downarrow$$

$$C^\cdot(\mathfrak{gl}_N(A))$$

In this example $A = \Omega^{0,\cdot}(X)$.

Then

$$\mathfrak{gl}_N(\Omega^{0,\cdot}(X)) =$$ fields of $\mathfrak{gl}_N$ hCS thy.

$\rightsquigarrow$

$$C^\cdot(\mathfrak{gl}_N(\Omega^{0,\cdot})) =$$ observables of hCS thy.

$$\uparrow$$

$$C^\cdot(\mathfrak{gl}_\infty(\Omega^{0,\cdot})) =$$ Large N limit.

$$\uparrow$$

$$Cyc^\cdot(\Omega^{0,\cdot})[-1]$$

HKR:

$$PV^\cdot_{hol}(X) \xrightarrow{\cong} Hoch^\cdot(\mathcal{O}_X)$$

} resolve

$$PV^{\cdot,\cdot}(X) \xrightarrow{\simeq} \mathrm{Hoch}^{\cdot}(\Omega^{0,\cdot}_X)$$

} cyclic

$$PV^{\cdot,\cdot}(X)[[u]] \longrightarrow \mathrm{Hoch}(\Omega^{0,\cdot}_X)[[u]]$$

$$\circlearrowright$$

$$\bar\partial + u\partial \qquad\qquad d_{Hoch} + uB$$

↑ Connes's B operator.

RHS is $Cyc^{\cdot}(\Omega^{0,\cdot}_X)$.

Abstractly

$$PV^{\cdot,\cdot}_X[[u]] \longrightarrow C^{\cdot}(gl_\infty(\Omega^{0,\cdot}_X))\,.$$

③ Branes in top^l string

B-model on $\mathbb{C}^n$. Brane is labeled

by cplx submflds. when

$$\mathbb{C}^k \subset \mathbb{C}^n$$

the fields on brane are

$$\mathrm{Ext}_{\mathcal{O}_{\mathbb{C}^n}}\left(\mathcal{O}_{\mathbb{C}^k}^{\oplus N}, \mathcal{O}_{\mathbb{C}^k}^{\oplus N}\right)$$

$$\simeq \Omega^{0,\cdot}(\mathbb{C}^k)[\varepsilon_1, \dots, \varepsilon_{n-k}] \otimes \mathfrak{gl}_N$$

When n is odd, this can be thought

of as hCS on

$$\mathbb{C}^{k|n-k}.$$