Overview of the twisted holography program

Dylan Butson

University of Oxford
Mathematical Institute

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Outline

The goal of this talk is to explain the general set-up for formulating twisted holography type conjectures:

Disclaimer: I am not an expert! All the glory, no accountability.

The input data required for the hypotheses of the meta-conjecture roughly speaking consists of two objects:

1. a 'string theory background' on an underlying manifold M,
2. a 'stack of N branes' in this background, along $L \subset M$.

The expected mathematical output data is:

1. an algebra A associated to the string theory background,
2. compatible algebras B_N associated to the stack of N branes,
3. compatible algebra maps $A^! \to B_N$ defined for each N.

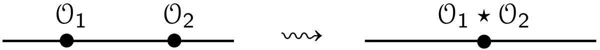
Conjecture: The twisted holographic principle [Costello-Li]
The induced map of algebras in the $N \to \infty$ limit

$$A^! \xrightarrow{\cong} B_\infty \qquad \text{is an isomorphism.}$$

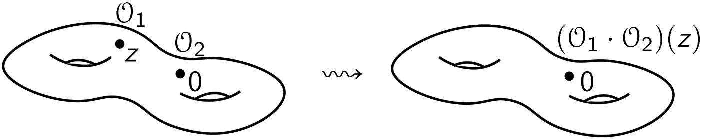
Algebras of observables in holomorphic-topological QFT

The 'algebras' in the statement of the conjecture are the quantum observables in holomorphic-topological QFTs (or string FTs).

In a 1d TFT, observables are a usual (associative) algebra:



In a 2d chiral CFT, observables are a vertex (or chiral) algebra:



Algebras of observables in holomorphic-topological QFT

In n dimensional TFTs, we have n compatible associative algebra structures, coming from the different directions to collide along.

Thus, the observables are an $\mathbb{E}_n$ (topological factorization) algebra.

See e.g. [L, AF, CG]

In higher dimensional holomorphic theories, the observables are higher dimensional holomorphic/algebraic factorization algebras.

See e.g. [BD, FG, CG, Williams, (B.)]

More generally, mixed holomorphic-topological theories on $X \times \mathbb{R}^n$ are factorization $\mathbb{E}_n$ algebras on X, or $\mathbb{E}_n$ algebras in $\mathrm{Alg}^{\mathrm{fact}}(X)$.

See e.g. [CG, GRW, Oh-Yagi, BBZBDN, R, BFN, B.]

However, it's difficult to actually produce such objects associated to a given field theory! Even more so with 'stringy corrections'...

In practice, we can do things systematically at the level of classical field theory, but must treat quantizations on a case-by-case basis.

Classical field theory

A classical field theory on M is given by a space of fields $\mathcal{F} = \Gamma(F)$ and an action functional $S \in \mathcal{O}_{\mathrm{loc}}(\mathcal{F}) = \mathrm{Sym}^{\bullet}_{\mathcal{O}_M}(\mathcal{J}(\mathcal{F})^{\vee}) \otimes_{\mathcal{O}_M} \Omega_M^{\mathrm{d}_M}$.
The derived space of solutions to the Euler-Lagrange equations modulo local gauge transformations is modeled by a local DGLA $\mathcal{L}$:

Example: Let $\mathcal{F} = C^\infty_M$ with action $S(\varphi) = \varphi(d * d\varphi) + \varphi B(\varphi, \varphi)$
Then $\mathcal{L} = C^\infty_M[-1] \xrightarrow{*d*} \Omega_M^{\mathrm{d}_M}[-2]$ with $[,] = B : C^\infty_M \otimes C^\infty_M \to \Omega_M^{\mathrm{d}_M}$

Example: Let $\mathcal{F} = \Omega^1_M \otimes \mathfrak{g}$ with $S(A) = \mathrm{Tr}(A \wedge dA + A \wedge [A, A])$ and infinitesimal gauge symmetry given by the action of $\Omega^0_M \otimes \mathfrak{g}$.
$$\mathcal{L} = \left[\Omega^0_M \otimes \mathfrak{g} \xrightarrow{d} \Omega^1_M \otimes \mathfrak{g}[-1] \xrightarrow{d} \Omega^2_M \otimes \mathfrak{g}[-2] \xrightarrow{d} \Omega^3_M \otimes \mathfrak{g}[-3]\right],$$
or more succinctly $\mathcal{L} = \Omega^{\bullet}_M \otimes \mathfrak{g}$, with bracket $[,]_{\mathcal{L}} = \wedge \otimes [,]_{\mathfrak{g}}$.

Example: Let $\mathcal{F} = \Omega^0_{\mathbb{R}} \otimes V$ with V symplectic, $S(\varphi) = \omega_V(\varphi, d\varphi)$
Then $\mathcal{L} = \left[\Omega^0_{\mathbb{R}} \otimes V[-1] \xrightarrow{d} \Omega^1_{\mathbb{R}} \otimes V[-2]\right] = \Omega^{\bullet}_{\mathbb{R}} \otimes V[-1]$.

Example: Let $\mathcal{F} = \Omega^0_{\mathbb{C}} \otimes V$ with $S(\varphi) = \omega_V(\varphi, \bar\partial\varphi)dz$
Then $\mathcal{L} = \left[\Omega^0_{\mathbb{C}} \otimes V[-1] \xrightarrow{\bar\partial} \Omega^{0,1}_{\mathbb{C}} \otimes V[-2]\right] = \Omega^{0,\bullet}_{\mathbb{C}} \otimes V[-1]$.

Open and closed (classical) string field theory

The input data for a twisted string theory background is a Calabi-Yau category $\mathcal{C}$, which should be 'geometric' and 'CY3'.

The main examples are topological Type II strings, denoted by $\mathbb{R}^{2k}_A \times \mathbb{C}^{5-k}_B$ corresponding to $\mathcal{C} = \mathsf{Fuk}(\mathbb{R}^{2k}) \otimes \mathsf{Coh}(\mathbb{C}^{5-k})$ 'mod 2'.

A 'brane' is an object $C \in \mathcal{C}$, and a stack of N is the object $\mathcal{C}^{\oplus N}$.

In Type IIA (when k is odd), the branes are products $L \times X$ for $L \subset \mathbb{R}^{2k}$ Lagrangian and $X \subset \mathbb{C}^{5-k}$ holomorphic.

These have odd dimension n, but are called D(n-1) branes.

Similarly, in Type IIB (k even), the D branes are even dimensional.

The classical open string field theory associated to a brane $C \in \mathcal{C}$ is described by the local DGLA $\mathcal{L} = \mathbb{R}\mathcal{H}om_{\mathcal{C}}(C, C)$.

Example A stack of N D3 branes on $\mathbb{R}^2 \times \mathbb{C} \subset \mathbb{R}^4 \times \mathbb{C}^3$ gives

$$\mathcal{L} = \mathbb{R}\mathcal{H}om_{\mathsf{Fuk}(T^\vee \mathbb{R}^2)}(\mathbb{R}^2, \mathbb{R}^2) \boxtimes \mathcal{H}om_{\mathsf{Coh}(\mathbb{C}^3)}(\mathcal{O}_{\mathbb{C}}^{\oplus N}, \mathcal{O}_{\mathbb{C}}^{\oplus N})$$
$$\cong \Omega^\bullet_{\mathbb{R}^2} \boxtimes \Omega^{0,\bullet}_{\mathbb{C}} \otimes \mathfrak{gl}_N[\varepsilon_1, \varepsilon_2] \text{ w. } |\varepsilon_i| = 1.$$

The classical closed string field theory associated to the twisted type II background is $\mathcal{L} = \mathsf{CC}^\bullet(\mathcal{C})[1] \cong \Omega^\bullet_{\mathbb{R}^{2k}} \otimes \mathsf{PV}^{\bullet,\bullet}_{\mathbb{C}^{d-k}}[[t]][1]$.

The Ω-background

The algebraic structures on quantizations of the open SFTs on higher dimensional branes are more subtle homotopically.

It's convenient to 'localize' to lower dimensional submanifolds via Ω-backgrounds, interpret as associative or vertex algebras.

My thesis explains this mechanism at the quantum level in algebraic terms. Paper about classical version (finally!) soon:

Example: Kapustin twist of 4d N=2 G gauge theory w matter V
$\mathcal{L} = \Omega^{\bullet}_{\mathbb{R}^2_\varepsilon} \boxtimes \Omega^{0,\bullet}_{\mathbb{C}} \otimes \mathfrak{g}_V \rightsquigarrow \Omega^{0,\bullet}_{\mathbb{C}} \otimes \mathfrak{g}_V \quad \text{w.} \quad \mathfrak{g}_V = \mathfrak{g} \ltimes V[-1] \ltimes \mathfrak{g}^*[-2]$
My thesis explains quantum version of this, gives CDOs to $[N/G]$.
Special case of 4d N=4 G gauge theory gives CDOs to $[\mathfrak{g}/G]$.

Example: Theory of D6s on $\mathbb{R}^3 \times \mathbb{C}^2 \subset \mathbb{R}^6 \times \mathbb{C}^2$ localizes to 5d CS
$\mathcal{L} = \Omega^{\bullet}_{\mathbb{R}^3_\varepsilon} \boxtimes \Omega^{0,\bullet}_{\mathbb{C}^2} \otimes \mathfrak{gl}_N \rightsquigarrow \Omega^{\bullet}_{\mathbb{R}} \boxtimes \Omega^{0,\bullet}_{\mathbb{C}^2} \otimes \mathfrak{gl}_N$

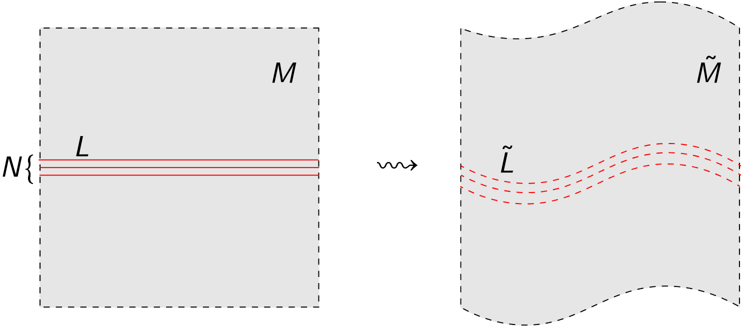
Example: Theory of D5s on $\mathbb{R}^4 \times \mathbb{C} \subset \mathbb{R}^8 \times \mathbb{C}$ localizes to 4d CS
$\mathcal{L} = \Omega^{\bullet}_{\mathbb{R}^4_\varepsilon} \boxtimes \Omega^{0,\bullet}_{\mathbb{C}} \otimes \mathfrak{gl}_N \rightsquigarrow \Omega^{\bullet}_{\mathbb{R}^2} \boxtimes \Omega^{0,\bullet}_{\mathbb{C}} \otimes \mathfrak{gl}_N$

The inputs for twisted holography

We said the inputs for the twisted holography setup are:

1. a 'string theory background' on an underlying manifold M,
2. a 'stack of N branes' in this background, along $L \subset M$.

The presence of the branes deforms the background $M \rightsquigarrow \tilde{M}$.

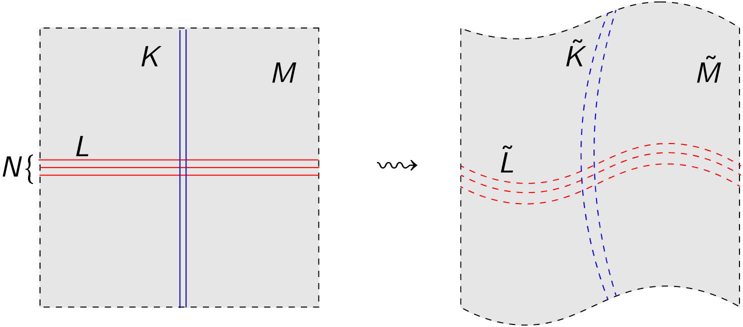


Conjecture: For A, B_N quantizations on $\tilde{M}$ and L, $(\iota^! A)^! \xrightarrow{\cong} B_\infty$.

Hopefully, $A^!$ equivalent to algebra of states at ∞ under Witten diagrams, via observables on transverse boundary condition at ∞.

Twisted holography with auxiliary branes

We can also consider the above setup with auxiliary branes. Naively, this seems more complicated, but in nice situations only the open string sectors contribute, so much more tractable! Also occurs naturally as pure geometry in other duality frames.



Conjecture: For A and B_N on $(\tilde{M}, \tilde{K})$ and L, $(\iota^! A)^! \xrightarrow{\cong} B_\infty$.

Or, ideally: For A and B_N on $\tilde{K}$ and $\tilde{K} \cap \tilde{L}$, $(\iota^! A)^! \xrightarrow{\cong} B_\infty$.

The example of M2 branes on the A_{k-1} singularity

Consider Type IIA on $M = (\mathbb{R}^4_{\varepsilon_1,\varepsilon_2} \times \mathbb{R}^2)_A \times \mathbb{C}^2_B \rightsquigarrow \mathbb{R}^2_A \times \mathbb{C}^2_B$

With N D2 branes along $L = \mathbb{R}^2_{\varepsilon_1} \times \mathbb{R} \times \{0\} \rightsquigarrow \mathbb{R} \times \{0\}$

Aux. k D6 branes along $K = \mathbb{R}^2_{\varepsilon_2} \times \mathbb{R} \times \mathbb{C}^2 \rightsquigarrow \mathbb{R} \times \mathbb{C}^2_B$

This comes from N M2 branes on an A_{k-1} singularity.

The 'gravity' theory localizes to 5d CS: $\mathcal{L}_{\mathrm{grav}} = \Omega^\bullet_{\mathbb{R}} \boxtimes \Omega^{0,\bullet}_{\mathbb{C}^2} \otimes \mathfrak{gl}_k$

The quantization A should be an $\mathbb{E}_1$ chiral algebra on $K = \mathbb{R} \times \mathbb{C}^2$

The gauge theory on N D2s localizes to $\mathcal{L}_{D2} = \Omega^\bullet_{\mathbb{R}} \otimes \mathfrak{gl}_N[\varepsilon_1, \varepsilon_2]$

The D2-D6 strings contribute $\mathcal{L} = \Omega^\bullet_{\mathbb{R}} \otimes T^\vee \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^N)[-1]$

$\mathcal{L}_{gauge} = \Omega^\bullet_{\mathbb{R}} \otimes (\mathfrak{gl}_N \ltimes (T^\vee \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^N) \oplus T^\vee \mathfrak{gl}_N)[-1] \ltimes \mathfrak{gl}_N^*[-2])$

The quantization B_N should be an $\mathbb{E}_1$ algebra along $L \cap K = \mathbb{R}$.

Theorem [Costello '17] Let A, B_N be quantizations of $\mathcal{L}_{\mathrm{grav}}, \mathcal{L}_{gauge}$

There is an isomorphism of associative algebras $(\iota^! A)^! \xrightarrow{\cong} B_\infty$.

Quantizations: methods and difficulties

To make the Theorem precise, we need to understand A, B_N, $A^!$.

CG gives very general abstract existence theorem for C^∞ factorization algebras of quantum observables, but no presentation.

Geometric quantization gives nice answers in the best cases:

Example: G-gauged topological mechanics to V gives quantum Hamiltonian reduction of $W_\hbar(V)$ by G; $\mathcal{O}_\hbar(M(k,N))$ for $\mathcal{L}_{gauge}$.

But geometric quantization of ClFT is an art, no general answer.

Dream: Compute quantum algebra structure on $C^\bullet_{\mathrm{CE}}(\mathcal{L})_x[\hbar]$ from Feynman diagrams; more robust but harder to make rigorous.

In our example, $C^\bullet_{\mathrm{CE}}(\mathcal{L}_{\mathrm{grav}})_x[\hbar] = C^\bullet_{\mathrm{CE}}(\mathfrak{gl}_k[z_1,z_2])[\hbar]$, with some deformation at $\hbar \neq 0$ from commutative to $\mathbb{E}_1$-chiral on $\mathbb{C}^2$.

Thus, $(\iota^! A)^! = \mathcal{U}(\mathfrak{gl}_k[z_1,z_2])[\hbar]$ with Koszul dual deformation.

Kevin's theorem is $\mathcal{U}(\mathfrak{gl}_k[z_1,z_2])[\hbar] \xrightarrow{\cong} \mathcal{O}_\hbar(M(k,N))$ as $N \to \infty$.

Four dimensional Chern-Simons and the D3-D5 system

Following IMZ, consider IIB on $M = (\mathbb{R}^4_{\varepsilon_1,\varepsilon_2} \times \mathbb{R}^4)_A \times \mathbb{C}_B$

With N D3 branes along $L = \mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}^2 \times \{0\} \rightsquigarrow \mathbb{R}^2 \times \{0\}$

Aux. k D5 branes along $K = \mathbb{R}^2_{\varepsilon_2} \times \mathbb{R}^2 \times \mathbb{C} \rightsquigarrow \mathbb{R}^2 \times \mathbb{C}$

The 'gravity' theory localizes to 4d CS: $\mathcal{L}_{\mathrm{grav}} = \Omega^\bullet_{\mathbb{R}^2} \boxtimes \Omega^{0,\bullet}_{\mathbb{C}} \otimes \mathfrak{gl}_k$

The quantization A should be an $\mathbb{E}_2$ chiral algebra on $K = \mathbb{R}^2 \times \mathbb{C}$

The gauge theory on N D3s localizes to $\mathcal{L}_{D3} = \Omega^\bullet_{\mathbb{R}^2} \otimes \mathfrak{gl}_N[\varepsilon]$

The D3-D5 strings give $\mathcal{L} = \Omega^\bullet_{\mathbb{R}} \otimes T^\vee(\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^N)[1])[-1]$

Thus, $\mathcal{L}_{gauge} = \Omega^\bullet_{\mathbb{R}} \otimes (\mathfrak{gl}_N[\varepsilon] \ltimes (T^\vee(\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^N)[1]))[-1])$

IMZ prove the quantization of $\mathcal{L}_{gauge}$ as $N \to \infty$ is $Y_\hbar(\mathfrak{gl}_k)$.

They also prove the Yangian arises from Witten diagrams.

One can also check $C^\bullet_{\mathrm{CE}}(\mathcal{L}_{\mathrm{grav}})_x[\hbar] = C^\bullet_{\mathrm{CE}}(\mathfrak{gl}_k[z])[\hbar]$ so that the Koszul dual $(\iota^! A)^!$ is deformation of $\mathcal{U}(\mathfrak{gl}_k[z])[\hbar]$.

It would be nice to understand why the latter two are the same.