TWISTED HOLOGRAPHY AND KOSZUL DUALITY

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1. From physical hologrpahy to Koszul duality

The first lecture was given by Kevin Costello.

1.1. **Motivation.** The most basic version of holography asserts the following

Conjecture 1.1 (Maldacena-Witten)**.** Type IIB string theory on $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ super Yang-Mills for $G = U(N)$ on S^4 in the limit where $N \to \infty$.

Here, AdS_5 refers to five dimensional Lorentzian hyperbolic space $\mathbb{H}^5$. The S^4 on the gauge theory side arises as the boundary of $\mathbb{H}^5$.

Above, "conjecture" is meant in a physical sense as neither side of the correspondence has been formulated mathematically. Even at a physical level of rigour, understanding the above statement has some key difficulties:

- The gauge theory side must be defined nonperturbatively.
- The $AdS_5 \times S^5$ background is a Ramond-Ramond background and there are technical obstructions to understanding string theory in such backgrounds.

The goal of this seminar is to look at *twists*, or supersymmetry protected subsectors, of both sides, and examine the duality there. Such twists are holomorphic-topological field theories. We can attach familiar mathematical objects such as $\mathbb{E}_n$ algebras, vertex algebras, categories, and more to such field theories and attempt to formulate holography mathematically in terms of such objects. For example:

- To twists of 4d $\mathcal{N} = 4$ theories, we can attach categories of boundary conditions of compactifications to two dimensions - for particular twists, these recover Geometric Langlands categories. We can also attach framed $\mathbb{E}_4$-algebras, and vertex algebras.
- Twists of type II string theory admit descriptions in terms of topological strings, which can be described by certain Calabi-Yau categories and invariants thereof.

Thus, we expect a mathematical codification of holography at the level of twists to uncover novel relationships between the above structures.

1.2. **Twisting Supersymmetric Field Theories.**

Definition 1.2. A supersymmetric quantum field theory in dimension n is a quantum field theory carries an action of the super poincare algebra $(\Pi S \oplus \mathbb{R}^n) \ltimes \mathfrak{so}(n)$.

Above S is a spin representation in n dimensions. There's also an action of the commutant $G_R \subset \mathrm{End}(S)$ of $\mathrm{Spin}(n)$, the so-called R-symmetry group.

Twisting consists of a two step procedure.

(1) Choose a subgroup $H \subset \mathrm{Spin}(n)$ and a homomorphism $\rho : H \to G_R$. Letting $\mathrm{SISO}(n)$ denote the group exponentiating the super-poincare algebra, we replace $\mathrm{SISO}(n) \times G_R$ with the twisted product $\mathrm{SISO}(n) \times_\rho G_R$. That is, we use the action of R-symmetry to change the spin of the fields. Locally, this is a very mild modification of the theory - we've decided to treat some of the scalars of the theory as e.g. components of a 1-form.

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(2) Choose a $Q \in S$ that is a scalar under the action of $\mathrm{SISO}(n) \times_\rho G_R$ such that $[Q, Q] = 0$ and add Q to the BRST diferential of the QFT. This typically simplifes the theory drastically - it renders the action of Q exact translations on the theory homotopically trivial.

If $Im[Q, -] = \mathbb{C}^n$ then all translations act trivially, and the result is a topological field theory. For reasons of supersymmetric linear algebra, $Im[Q, -]$ is always coisotropic. If it is of the minimal dimension $\lceil n/2 \rceil$, then the twist is holomorphic. In general we find a theory that is a mixture of holomorphic and topological.

More homotopically, a choice of Q gives an action of an abelian superalgebra $\Pi\mathbb{C}$ on the theory. Adding Q to the differential is a model for the derived invariants for this action. The result is a module over $\mathcal{O}(B\Pi\mathbb{C}) = \mathbb{C}[\![t]\!]$. If we have a $\mathbb{C}^\times$ action where Q has weight 1, then we have a Rees family. The twisted theory lives over the generic point.

1.2.1. *The Ω-deformation.* A variant of this construction involves taking instead a supercharge Q that squares to an infinitessimal rotation. We can then perform an equivariant cohomology construction - we can take Q cohomology at the S^1 fixed points. If a theory on $\mathbb{R}^2 \times \mathbb{R}^2 \times \cdots$ is deformed by a supercharge Q whoes square rotates the two planes with speeds $\varepsilon_1, \varepsilon_2$, we will denote the spacetime by $\mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}^2_{\varepsilon_2}$.

Example 1.3. There is a holomorphic-topological twist of 4d $\mathcal{N} = 4$ super Yang-Mills on a product of Riemann surfaces $\Sigma \times C$ which is topological on Σ and holomorphic on C. This twist was first studied by Kapustin, and provides a home for some familiar objects in geometric representation theory.

Indeed, compactifying the theory on C yields a 2d TQFT whose category of boundary conditions is $\mathrm{Coh}(\mathrm{Higgs}_G C)$. This category featuers in the conjectural Dolbeault Geometric Langlands which posits an equivalence of categories

$$\mathrm{Coh}(\mathrm{Higgs}_G C) \cong \mathrm{Coh}(\mathrm{Higgs}_{\check{G}} C).$$

Let's specialize to the case $\Sigma = \mathbb{R}^2$ and $C = \mathbb{C}$. We may subject the theory to an Ω-deformation along $\mathbb{R}^2$, to get a holomorphic field theory on $\mathbb{C}$. The algebra of observables of the result is a chihral algebra on $\mathbb{C}$. Explicitly, it is given by the BRST reduction of chiral differential operators on $\mathfrak{g}$ by G acting by the adjoint action, or equivalently chiral differential operators on the quotient stack $\mathfrak{g}/G$.

Even more explicitly, the VOA is given by the BRST reduction of the VOA generated by fields X_a, Y_a where a is a lie algebra index, with OPE

$$X_a(0)Y_a(z) \sim \delta_{ab}\frac{1}{z}.$$

This example fits in the larger program of [] of attaching VOAs to 4d $\mathcal{N} = 2$ superconformal field theories; this is the output of their construction applied to an $\mathcal{N} = 4$ theory.

Example 1.4. Next we consider a topologically twisted Ω-deformed version of the worldvolume theory of M2 branes. The resulting theory is succinctly described as ADHM quantum mechanics. This is the theory whose algebra of operators is the quantum Hamiltonian reduction of $T^*(\mathfrak{gl}_N \oplus \mathbb{C}^N)$ by GL_N. This algebra is also known as the spherical double affine hecke algebra, which is a deformation quantization of $Hilb^N(\mathbb{C}^2)$. This can also be thought of as the output of the Coulomb branch construction of Braverman-Finkelberg-Nakajima [] for the ADHM quiver, where are we using that this quiver is self 3d-mirror.

Thus in both examples, twisting and Ω-deformation allows us to extract familliar objects of geometric representation theory from supersymmetric field theories. Our goal will be to realize these objects in the large N limit in terms of something gravitational nature.

1.3. **Twisting Supergravity.** In supergravity, the supersymmetries are gauged. The fields are a quotient stack where we quotient by the action of a huge supergroup rougly given by $C^\infty(\mathbb{R}^n, \mathrm{SISO}(n))$. Thus, the fields of the theory live over the space of maps $R^n \to B\mathrm{SISO}(n)$; this classifying stack is the analog of the moduli of vacua in gauge theory.

Recall that for a lie algebra $\mathfrak{g}$, we can think of the classifying stack $B\mathfrak{g}$ as the space of solutions to the Maurer-Cartan equation in $\mathfrak{g}$, modulo gauge. The set of solutions to the Maurer-Cartan equation in the supertranslation algebra $\mathbb{R}^n \oplus \Pi S$ is simply the collection of square zero supercharges, and is called the *nilpotence variety* []. This variety is closely related to classical objects such as the space of pure spinors studied by Cartan.

We conclude that supergravity contains a field that looks like a map

$$c_\alpha : \mathbb{R}^n \to \text{ nilpotence variety } /\mathrm{Spin}(n).$$

This is the ghost for gauged supersymmetries - since supersymmetries are fermionic, this ghost is bosonic.

Definition 1.5 (Costello-Li)**.** Twisted supergravity is supergravity in a vacuum where the bosonic ghost takes a nonzero VEV.

That is, twisted supergravity is simply a vacuum for ordinary supergravity.

To see how this simplifies the theory, note that there are terms in the supergravity action of the form $c_\alpha g \psi^*$ where g is the metric and ψ^* is the antifield to the gravitino. This term enforces gauge invariance under local supersymmetry transformations that mix the metric and the gravitino. If c has a VEV this gives a mass term allowing us to integrate out certain components of the metric and gravitino. Alternatively, if c_{alpha} has a mass, the above term then generates a new differential in the BRST complex which leads to cohomological cancellations.

Note that a priori this procedure seems very different from twisting a supersymmetric field theory. We can reformulate twisting a supersymmetric field theory to similarly to the above. We can enlarge the space of fields of a gauge theory by adding global supersymmetries as ghosts by hand, and turn on a constant value for these ghosts.

A key feature of twisted supergravity is that worldvolume theories of branes in twisted supregravity backgrounds are naturally twists of the supersymmetric field theories one finds as worldvolume theories in the physical string. The examples in the previous subsection will arise from worldvolume theories of branes in suitable twisted supergravity backgrounds.

Example 1.6. In the early 90s, it was discovered that the topological string on a CY3 embeds in the type II string. Originally, this was expressed by saying that certain quantities in the type II string can be computed in the topological string. That is, the topological string computes type II string amplitude in the presence of the so-called self-dual graviphoton background. In modern language, this can be expressed by saying that the topological string arises from Ω-deforming a twist of the string.

More precisely, the IIB string on $\mathbb{R}^2_\varepsilon \times \mathbb{R}^2_{-\varepsilon} \times X$ where X is a CY3 gives the B-model on X. Similarly, the IIA string on $\mathbb{R}^2_\varepsilon \times \mathbb{R}^2_{-\varepsilon} \times X$ gives the A-model on X.

Example 1.3 arises from considering a stack of D3 branes in the above Ω-deformed twist of IIB, wrapping $\mathbb{R}^2_{-\varepsilon} \times C$ where $C \subset X$ is a holomorphic curve.

Example 1.7. We can also consider an Ω-deformation of a $G2 \times \mathrm{SU}(2)$ twist of 11d supergravity; we denote the relevant background by $\mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}^2_{\varepsilon_2} \times \mathbb{R}^2_{\varepsilon_3} \times \mathbb{R} \times \mathbb{C}^2$ where $\sum \varepsilon_i = 0$.

It turns out that this can be described as a 5d noncommutative Chern-Simons theoy that is partially holomorphic and topological. The fundamental field is a partial gauge field

$$A \in \Omega^1(\mathbb{R} \times \mathbb{C}^2)/(dz_1, dz_2).$$

The action is given by

$$\frac{1}{\varepsilon_3} \int dz_1 dz_2 \frac{1}{2} A dA + \frac{1}{3} A * A * A$$

where $*$ denotes the Moyal product.

It is natural to wonder in what sense this describes a gravitational theory. Expanding the action gives

$$\int dz_1 dz_2 A dA + A \partial_{z_1} A \partial_{z_2} A + \cdots .$$

Let's analyze just the written terms. On $\mathbb{C}^2$ a deformation of the canonical holomorphic symplectic structure is given by $A \in \Omega^{0,1}(\mathbb{C}^2)$ satisfying the equation

$$\bar{\partial} A + \frac{1}{2} \varepsilon_{ij} \partial_{z_i} A \partial_{z_j} A = 0$$

. This equation is the Maurer-Cartan equation for the Beltrami differential $\varepsilon^{ij} \partial_{z_i} A \partial_{z_j}$. Expanding the action in components of the gauge field gives

$$\int dz_1 dz_2 A_t dA_{\bar{z}_i} + A_t \partial_{z_k} A_{\bar{z}_i} \partial_{z_l} A_{\bar{z}_j} + \cdots .$$

Varying with respect to A_t yields exactly the above Maurer-Cartan equation. Thus, we find that solutions to the equation of motion describe $\mathbb{R}$-families of holomorphic symplectic structures on $\mathbb{C}^2$. This is some particular class of deformations of a metric with SU(2) holonomy.

Example 1.4 arises as the worldvolume theory of M2 branes wrapping $\mathbb{R}^2_{\varepsilon_i} \times \mathbb{R}$.

Thus, we wish to make the following matches.

- The large N limit of the spherical DAHA and quantities in 5d noncommutative Chern-Simons
- Chiral differential operators on the adjoint quotient stack $\mathfrak{g}/G$ and quantities in the IIB string on $\mathbb{R}^2_\varepsilon \times \mathbb{R}^2_{-\varepsilon} \times X$.

1.4. **Global Symmetry Algebras.** The relation we are after is a litle involved, involving koszul duality of boundary operators. As a first consistency check though, we can extract certain lie algebras from both sides and check that they match. In physical AdS/CFT this is very simple.

The gravitational lie algebra $\mathfrak{g}_{grav}$ consists of gauge transformations that fix the metric. These are precisely isometries of $AdS_5 \times S^5$ some ferminoic symmetries. These will match with superconformal symmetries in the gauge theory. Explicitly, the isometries of $AdS_5 \times S^5$ are given by $\mathrm{SO}(6) \times \mathrm{SO}(4,2)$ The first factor is the R-symmetry of 4d $\mathcal{N} = 4$ and the second factor is the conformal symmetries of S^4.

However, in the twisted setting, these algebras all receive infinite dimensional enhancements, and such a check carries more content.

Example 1.8. Let's work in the setting of example 1.7 and consider the gauge transformations of 5d Chern-Simons that fix the zero gauge field. The gauge transformations act by

$$A \mapsto A + d\chi + \bar{\partial}\chi + [\chi, A]$$

where the last term denotes the Moyal commutator. Therefore, the gauge transformations that preserve the zero gauge field consist of holomorphic functions on $\mathbb{C}^2$ with the Moyal commutator. Equivalently, this is the algebra of differential operators on $\mathbb{C}$, Diff($\mathbb{C}$). This lie algebra is a small deformation of the algebra of hamiltonian diffeomorphisms of $\mathbb{C}^2$, which is the so-called w_∞ algebra.

Let's now try and arrive at this algebra from ADHM quantum mechanics. We wish to compute the quantum hamiltonian reduction of $T^*(\mathfrak{gl}_N \oplus \mathbb{C}N)$ in the large N limit. We begin by describing functions on the classical hamiltonian reduction. To do so, we wish to look at level sets of the moment map and quotient by gauge. Let us choose coordinates $I_i \in \mathbb{C}^N$, $J^i \in (\mathbb{C}^N)^*$, $X^i_j, Y^i_j \in \mathfrak{gl}_N$. The moment map equation is given by $[X, Y] + IJ = c$ where c is some generic element of $\mathfrak{gl}_N$.

Lemma 1.9. The algebra of functions on the classical hamiltonian reduction is generated by monomials of the form IX^rY^sJ. These generators are algebraically independent if $r + s < N$.

Note that these monomials are clearly GL_N invariant - the function takes a vector, acts on it by a bunch of matrices, and then pairs with a covector. The fact that these expressions generate the algebra is a consequence of the moment map relation and classical invariant theory. The independence of these generators is trickier.

A consequence of the constraint for independence is that in the large N limit, all generators are independent. Thus, we find that functions on the symplectic quotient are given by $\mathrm{Sym}(\mathcal{O}(\mathbb{C}^2))$, under the identification $IX^rY^sJ \mapsto z_1^r z_2^s$.

The quantum hamiltonian reduction will be a flat deformation of the above. The result will be a quantum deformation of $U(\mathrm{Diff}\mathbb{C})$. To see this we stipulate that the coordinates X, Y, I, J are now elements of a Weyl algebra:

$$[X^i_j, Y^k_l] = \hbar\delta^i_j\delta^k_l, \qquad [I_i, J^j] = \hbar\delta^j_i.$$

We now wish to compute commutators between the same monomials we found previously.

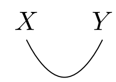
Write each monomial in index-ful notation as

$$IX^rY^sJ = I_{i_0}X^{i_0}_{i_1}X^{i_1}_{i_2}\cdots Y^{i_{r+s}}_{i_{r+s+1}}J^{i_{r+s+1}}.$$

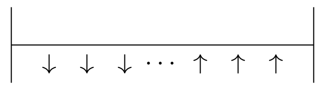
The above commutation relations stipulate that

$$X^i_j = \hbar\frac{\partial}{\partial Y^j_i}, \qquad I_i = \hbar\frac{\partial}{\partial J^i}.$$

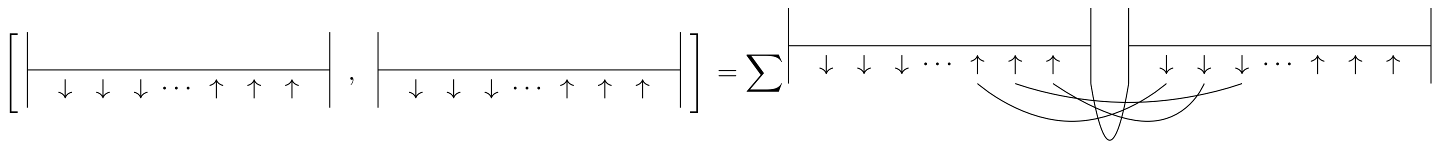
Let's introduce the diagrammatic notation



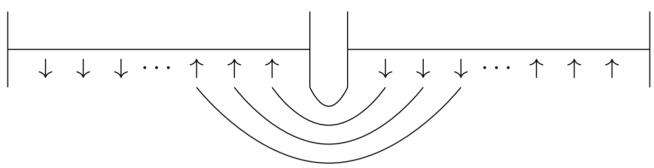
to denote the commutator between X and Y which yields a factor of $\hbar$. We diagrammatically write the monomial IX^rY^sJ as



where there are r up arrows depicting the factors of X and s down arrows depicting the factors of Y. In terms of this notation, the commutator is expressed diagrammatically as



where the sum is over all possible contractions. The right hand-side of the expression can then be simplified using the moment map relation. However, the dominant term in the large N limit is simple - it comes from the diagram where all the adjacent indices are contracted.



This recovers precisely the Lie bracket on Diff($\mathbb{C}$). Thus, we arrive at the statement that the large N quantum Hamiltonian reduction of the ADHM quiver is UDiff($\mathbb{C}$). More precisely, we find the central quotient of UDiff($\mathbb{C}$) where the identity operator in Diff($\mathbb{C}$) is identified with N. Indeed, this follows from taking the trace of the moment map relation $JI + [X, Y] = 1$.

1.5. **Backreaction.** A key aspect of physical holography that has not appeared in our twisted story so far is the appearance of geometries like $AdS_m \times S^n$. Let's recall how this arises in the physical picture. This involves the following steps

(1) Consider some supergravity theory on $\mathbb{R}^n$ with a brane on $\mathbb{R}^d \subset \mathbb{R}^n$. The brane deforms the action to first order by the inclusion of a source term - this is some curved deformation of the L_∞ algebra describing the gravitational theory.
(2) Solve the equations of motion in the presence of this source term; a solution is called the field sourced by the brane. Among such solutions is a metric that is singular along the locus of the brane $\mathbb{R}^d$. The complement of the brane $\mathbb{R}^n \setminus \mathbb{R}^d$ with this metric is called the *black brane geometry*.
(3) Take the *near horizon limit*. This involves zooming in near the location of the brane. The result is $AdS_{d+1} \times S^{n-d-1}$.

We can repeat this proceudre in the twisted theory. The last step won't be necessary in the twisted setting as the theories will always be topological in the radial direction transverse to the brane.

Example 1.10. Let's spell out the above in the 5d chern-simons theory of examples 1.4, 1.7. We consider a stack of $NM2$ branes along the topological direction; this introduces the source term $N\int_{\mathbb{R}} A$ to the action. The equation of motion in the presence of this term, to linear order, simply reads $(d + \bar{\partial})A = N\delta_{\mathbb{R}}$. This is solved by the Bochner-Martinelli kernel

$$A = N\frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{\|z\|^4}$$

. Thus, the complement of the brane, $\mathbb{R} \times (\mathbb{C}^2 \setminus \{0\})$, with this closed string field is our analog of $AdS_2 \times S^3$.

We now wish to study boundary operators for the compactification of the theory on $S^3 \subset \mathbb{C}^2 \setminus \{0\}$. The effective 2d theory obtained by compactification is 2d BF theory for the lie algebra Diff($\mathbb{C}$), on the half space

$\mathbb{R} \times \mathbb{R}_{\geq 0}$. Here, $R_{\geq 0}$ is the radial direction in $\mathbb{C}^2 \setminus \{0\}$; let us choose a coordinate r. The algebra we wish to study is the algebra of boundary operators for a Dirichlet boundary condition at $r = \infty$. It is easy to see that the algebra of tree level boundary operators for BF theory with lie algebra $\mathfrak{g}$ is the universal enveloping algebra $U(\mathfrak{g})$. Alternatively, we can view the BF theory as a Poisson σ-model and appeal to Kontsevich's work on deformation quantization.

Thus in this example, we find (at least at tree level) that the algebra of operators on a large number of M2 branes, which we computed in example 1.8 agrees with boundary local operators in the compactification of supergavity on a sphere linking the branes. This is what is expected from physical AdS/CFT.

Recall that in 1.8 we actually found a central quotient of $U(\mathrm{Diff}(\mathbb{C}))$. On the gravitational side, this is a consequence of the backreaction.

In the previous example, the effect of the backreaction was very mild. In other examples, it will give a central extension of the gravitational algebra, rather than a central quotient. Physically, the difference stems from the following two approaches to the backreaction:

(1) Take the mode corresponding to the backreaction as dynamical. This happens when we consider branes electrically coupled to the gravitational theory. Then the boundary algebra will have a ceentral element corresponding to N.
(2) Take the mode corresponding to the backreaction to be non dynamical. This happens when we consider branes magnetically coupled to the gravitational theory. In this case, N enters as an extra parameter, i.e. the algebra of operators gets an N-dependent central extension.

Example 1.11. Let's work in the example of 1.6. Consider the B-model on $\mathbb{C}^3$ with a stack of N branes along $\mathbb{C}$. For concreteness, let's fix coordinates z, w_1, w_2 and take the brane to wrap the $w_i = 0$ plane. The fundamental field of the theory is a Beltrami differential

$$\mu \in \Omega^{0,1}(\mathbb{C}, T\mathbb{C}^3) \cong \Omega^{2,1}(\mathbb{C}^3)$$

where the last identity uses the Calabi-Yau structure. The inclusion of the brane deforms the action to leading order by a term of the form $N \int_{\mathbb{C}} \partial^{-1}\mu$. This is a magnetic coupling, and can be thought of as a Wess-Zumino type term which tells us to integrate μ over a three cycle whose boundary is the brane $\mathbb{C}$. Solutions to the equation of motion in this background are given by

$$\mu = N \frac{\bar{w}_1 d\bar{w}_2 - \bar{w}_2 d\bar{w}_1}{\|w\|^4} \partial_z.$$

Lemma 1.12 (Costello-Gaiotto)**.** $\mathbb{C} \times (\mathbb{C}^2 \setminus \{0\})$ deformed by this Beltrami differential is $\mathrm{SL}_2\mathbb{C}$.

Proof. We wish to determine functions that are holomorphic with respect to the deformed complex structure and show that they satisfy the relations of the algebra of functions on $\mathrm{SL}_2\mathbb{C}$. Since the beltrami differential only has a ∂_z component, the functions w_1, w_2 are still holomorphic. The function z is no longer holomorphic, but the functions

$$v_1 = zw_1 - N\frac{\bar{w}_2}{\|w\|^2}, \qquad v_2 = zw_2 + N\frac{\bar{w}_1}{\|w\|^2}$$

are. These coordinates collectively satisfy

$$v_2 w_1 - v_1 w_2 = N.$$

□

As in the previous example we can try to match boundary operators for this gravitational theory with large N chiral differential operators on the adjoint quotient stack. In [] Costello and Gaiotto compare the global symmetry algebras of the theories.

The global symmetry algebra $\mathfrak{g}_{grav}$ of the B-model on $\mathrm{SL}_2\mathbb{C}$ is generated by

$$X \in \mathrm{Vect}_0(\mathrm{SL}_2\mathbb{C}), \qquad f, g \in \Pi\mathcal{O}(\mathrm{SL}_2(\mathbb{C})).$$

Here $\mathrm{Vect}_0(\mathrm{SL}_2\mathbb{C})$ denotes holomorphic divergence free vector fields. The lie structure involves three brackets. Two divergence vector fields bracket by the lie derivative, and a divergence free vector field acts on a function by differentiation. There is a natural bracket on the functions by saying that

$$[f, g] = (\partial f \wedge \partial g) \vee \Omega^{-1}$$

where Ω^{-1} denoets the inverse of the Calabi-Yau volume form.

Let A_N denote the mode algebra of chiral differential operators on the quotient stack $\mathfrak{gl}_N/\mathrm{GL}_N$.

Theorem 1.13 (Costello-Gaiotto)**.** As $N \to \infty$ there is an embedding $U\mathfrak{g}_{grav} \hookrightarrow A_\infty$. The image of $\mathfrak{g}_{grav}$ under this embedding consists of single trace modes preserving the vacuum at 0 and ∞.

The backreaction is essential in establishing this result. In this example, the coupling of branes is magnetic, so we cannot treat the backreaction as dynamical.

Note that in this example the universal enveloping algebra of the gravitational symmetry algebra is much smaller than the algebra of modes for the gauge theory local operators. This is expected, the latter should be the same size as the mode algebra of boundary local operators in supergravity, and $U\mathfrak{g}_{grav}$ is much smaller than this. It was in a sense a coincidence that in example 1.8 $U\mathfrak{g}_{grav}$ recovered the entire algebra of boundary operators in 5d Chern-Simons.

2. A user's guide to holography

The second talk was given by Brian Williams. Our goal is to give a mathematical formulation of holography in terms of algebras of local operators.

2.1. Local Operators.

Let's consider a translation invariant field theory on $\mathbb{R}^n$ with fundamental field $\phi \in C^\infty(\mathbb{R}^n)$.

Definition 2.1. A linear local operator at $0 \in \mathbb{R}^n$ is given by

$$\phi \mapsto \partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} \phi(0),$$

where $k_j \geq 0$

That is, these are certain sections of the dual of the jet bundle of $\mathbb{R}^n$. We wish to consider the free commutative algebra generated by these. For us the above notion is definitional, but there is a more principled way to arrive at this starting from the notion of a factorization algebra. We will not take this perspective today.

In general, the field ϕ must satisfy some equations of motion - this will render some of the above operators trivial.

Example 2.2. Suppose $n = 1$ and we have the equation of motion $\partial_{x^2}\phi = 0$. Then the only nontrivial local operators are

$$\phi \mapsto \phi(0)$$
$$\phi \mapsto \partial_x \phi(0)$$

Moreover, there could be gauge symetries, in which case not all operators are invariant.

Example 2.3. Let $\mathfrak{g}$ be a lie algebra, and suppose we have a theory with fundamental field

$$A \in \Omega^1(\mathbb{R}^2, \mathfrak{g})$$

subject to the equation of motion

$$dA + \frac{1}{2}[A, A] = 0.$$

There is a gauge symmetry that acts on the fields by

$$A \mapsto A + dc + [c, A],$$

where $c \in C^\infty(\mathbb{R}^2, \mathfrak{g})$. We will impose the condition that local oeprators be invariant under this gauge symmetry homologically.

Let

$$\Omega^\bullet(\mathbb{R}^2) = \Omega^0 \otimes \mathfrak{g} \to \Omega^1 \otimes \mathfrak{g} \to \Omega^2 \otimes \mathfrak{g}.$$

This cochain complex has a natural structure of a dg lie algebra with lie bracket given by

$$[\alpha \otimes X, \beta \otimes Y] = (\alpha \wedge \beta)[X, Y]_{\mathfrak{g}}.$$

The solutions to the Maurer-Cartan equation for this dg lie algebra describe flat G-bundles up to gauge transformations. Really we have a local dg lie algebra, the dg lie structure naturally lives on the sheaf of sections of some graded vector bundle, and the dg lie structure is given by polydifferential operators.

Let's study the observables on a small disk $D \subset \mathbb{R}^2$. We can define this to be the Chevalley-Eilenberg cochains of the value of the above local dg lie algebra on a disk

$$C^\bullet(\Omega^\bullet(D) \otimes \mathfrak{g}).$$

Note that we can replace D with any open set, and this gadget carries the structure of a factorization algebra. We will not need the entirety of this structure today, but it is useful to mention that it is a cosheaf on $\mathbb{R}^2$. In particular, this means that for any inclusion of discs $D \subset D'$, we have a map

$$C^\bullet(\Omega^\bullet(D) \otimes \mathfrak{g}) \to C^\bullet(\Omega^\bullet(D')\mathfrak{g}).$$

The local operators at 0 are given by the limit

$$\lim_{D \ni 0} C^\bullet(\Omega^\bullet(D) \otimes \mathfrak{g}).$$

Let's try and compute this limit explicitly. Every term in the above diagram is in fact quasi-isomorphic. For any open disc D, the inclusion $\mathfrak{g} \hookrightarrow \Omega^\bullet(D) \otimes \mathfrak{g}$ as constant $\mathfrak{g}$-valued functions is a quasi-isomorpshim by the Poincare lemma. Thus, we find that local operators are given by $C^\bullet(\mathfrak{g})$. This cochain complex encodes the gauge symmetry. We have gotten rid of the linear gauge symmetry by way of the above quasi-isomorphism, but the nonlinear term in the gauge symmetry survives and is encoded by the Chevalley-Eilenberg differential.

What about quantum observables? These should deform the above and give rise to various kinds of noncommutative algebra. Some known examples:

(1) The operators of a 1d theory (quantum mechanics) form a dg/A_∞ algebra.
(2) The operators of a 2d chiral theory form a vertex algebra
(3) The local operators of a n dimensional topological field theory on $\mathbb{R}^n$ form an $\mathbb{E}_n$-algebra.

2.2. **Koszul duality.** An essential ingredient in twisted holography is koszul duality for local operators. Let's begin by discussing the case of associative algebras. Let A be a dg associative algebra and let $\varepsilon : A \to \mathbb{C}$ be an augmentation. The augmentation of course naturally gives $\mathbb{C}$ the structure of an A-module.

Koszul duality will be a statement about the dg category $A - mod$. Recall that this is a category where the homomorphisms between two A-modules M, N is given by $\mathbb{R}\mathrm{Hom}_A(M, N)$, which is any cochain complex whose cohomology groups are given by $\mathrm{Ext}_A(M, N)$. This can be computed by way of a projective resolution $\tilde{M}^\bullet$ of M:

$$\mathbb{R}\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_A(\tilde{M}^\bullet, N).$$

Definition 2.4. If A, ε is an augmented dg algebra, let $A^! = \mathbb{R}\mathrm{Hom}_A(\mathbb{C}, \mathbb{C})$. This naturally has the structure of a dg algebra. Call $A^!$ the Koszul dual of A

Note that the augmentation ε is an additional choice we have made in defining the Koszul dual.

Example 2.5 (Symmetric-Exterior Koszul duality)**.** Let $A = \mathbb{C}[x]$ with augmentation given by $\varepsilon(f) = f(0)$. Let's compute the Koszul dual. To do so, we wish to find a free resolution of $\mathbb{C}$. A natural resolution is given by the Koszul complex

$$\mathbb{C}[x] \to \mathbb{C}[x]$$

with differential given by multiplying by x. Alternatively we can rewrite this as $\mathbb{C}[x, \xi], d = x\partial_\xi$ where $|\xi| = -1$.

Thus we have that

$$\mathbb{R}Hom_{\mathbb{C}[x]}(\mathbb{C}, \mathbb{C}) \cong \mathrm{End}_{\mathbb{C}[x]}(\mathbb{C}[x, \xi]).$$

The latter endomorphism algebra is given by

$$(\mathbb{C}[x, \xi, \partial_\xi], d = [x\partial_\varepsilon, -])$$

where $|\partial_\xi| = 1$. Note that the inclusion of the subcomplex $(\mathbb{C}[\partial_\xi], d = 0)$ is a quasi-isomorphism. Thus, we have shown that $\mathbb{C}[x]^! = \mathbb{C}[\varepsilon]$ where $|\varepsilon| = 1$.

More generally, for a vector space V, we see that

$$\mathrm{Sym}V^! \cong \mathrm{Sym}(V^*[-1]).$$

Example 2.6. Let $\mathfrak{g} = \mathrm{Lie}(G)$. Recall the following classic result

Theorem 2.7. There is an isomorphism of cochain complexes $C^\bullet(\mathfrak{g}) \cong \Omega^\bullet(G)^G$ between the Chevalley-Eilenberg complex and the complex of left invariant differential forms on G with the deRham differential.

We will use this characterization tocompute the Koszul dual of $C^\bullet(\mathfrak{g})$. Note that for any $U \subset G$, since TG is trivializable, we can write $\Omega^\bullet(U) \cong C^\infty(U) \otimes \wedge^\bullet g^*$.

If $\hat{G}$ denotes a contractible neighborhood of $1 \in G$, the Poincare lemma tells us that $\mathbb{C} \hookrightarrow \Omega^\bullet(\hat{G})$ is a quasi-isomorphism. Moreover, the inclusion is a map of $\Omega^\bullet(G)^G$ modules, where the action is by wedge product. Thus, we see that $\Omega^\bullet(\hat{G})$ is a free resolution of the trivial $C^\bullet(\mathfrak{g})$ module.

This allows us to compute

$$C^\bullet(\mathfrak{g})^! = \mathrm{End}_{\Omega^\bullet(G)^G}(\Omega^\bullet(\hat{G})).$$

For every $X \in \mathfrak{g}$, lie differentiation by the infinitessimal left translation generated by this element, $\mathcal{L}_X \in \mathrm{End}_{\Omega^\bullet(G)^G}(\Omega^\bullet(\hat{G}))$. We therefore have a map from the free algebra on the $\mathcal{L}_X$ to $\mathrm{End}_{\Omega^\bullet(G)^G}(\Omega^\bullet(\hat{G}))$. Moreover, these endomorphisms satisfy the relation $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ so the map factors through the universal enveloping algebra. Thus, we have that $U\mathfrak{g} \hookrightarrow C^\bullet(\mathfrak{g})^!$. We can show that this inclusion is in fact a quasi-isomorphism.

2.3. **Coupling.** Now what does any of this have to do with holography? Let's try and illustrate this in a very simple toy model. Consider some theory of gravity on $\mathbb{R} \times \mathbb{R}^n$ with $\mathcal{A}$ the algebra of local operators. For us this is in particular a cochain complex, and typically in theories of gravity, there will be no cohomology in degree 0. Let's consider "a stack of N branes" on $\mathbb{R} \times \{0\}$. There is additional data involved in specifying the theory on the stack of branes, but it in particular gives a line defect in the gravitational theory. Let $\mathcal{B}(N, \cdots)$ denote the algebra of local operators for the theory on the brane.

Proposition 2.8. There is a canonical map of algebras

$$\phi_{N,\cdots} : \mathcal{A}^! \to \mathcal{B}(N, \cdots)$$

that becomes an equivalence in the large N limit.

The appearance of Koszul duality in the above stems from its role in coupling the theory on the stack of branes with the gravitational theory. Let's try and isolate this notion and develop it further. Suppose we have two theories with algebras of operators $\mathcal{A}$ and $\mathcal{B}$ and we wish to couple them. For simplicity, let us suppose that both $\mathcal{A}$ and $\mathcal{B}$ are dg algebras. What additional data needs to be specified?

Definition 2.9. An algebraic coupling is a deformation of the differential on $\mathcal{A} \otimes \mathcal{B}$ to $Q_A + Q_B + \tilde{Q}$ where

$$\tilde{Q} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$$

satisfying:

(1) $\tilde{Q}$ is a derivation
(2) $(Q_\mathcal{A} + Q_\mathcal{B} + \tilde{Q})^2 = 0$.

We will further assume that $\tilde{Q} = [\alpha, -]$ where $\alpha \in \mathcal{A} \otimes \mathcal{B}$ is of degree 1. We can think of this condition as saying that the coupling in fact comes from some local interaction. The second condition above implies that α satisfies

$$Q_\mathcal{A}\alpha + Q_\mathcal{B}\alpha + [\alpha, \alpha] = 0.$$

This tells us that α is a Maurer-Cartan element in the dg Lie algebra $\mathcal{A} \otimes \mathcal{B}$.

Theorem 2.10. Suppose $\varepsilon : \mathcal{A} \to \mathbb{C}$ is an augmentation. Then there is a 1-1 correspondence

- Maurer-Cartan elements $\alpha \in \mathcal{A} \otimes \mathcal{B}$ such that $(\varepsilon \otimes 1)(\alpha) = 0$
- Algebra homomorphisms $\phi_\alpha : \mathcal{A}^! \to \mathcal{B}$.

Example 2.11. Let $\mathcal{A} = C^\bullet(\mathfrak{g})$ with the augmentation given by projection onto the Sym^0 component. Let us suppose $\mathcal{B}$ has zero differential. An $\alpha \in C^\bullet(\mathfrak{g}) \otimes \mathcal{B}$ satisfying the Maurer-Cartan equation must in particular be of degree 1, so must live in $\mathfrak{g}^* \otimes \mathcal{B}$. This defines a linear map $\phi_\alpha : \mathfrak{g} \to \mathcal{B}$. The Maurer-Cartan equation is equivalent to saying that for all $x, y \in \mathfrak{g}$, ϕ_α satisfies

$$\phi_\alpha([x,y]) = \phi_\alpha(x)\phi_\alpha(y) - \phi_\alpha(y)\phi_\alpha(x).$$

That is, ϕ_α is a lie map. By universality, this lifts to a unique map of associative algebras $U\mathfrak{g} \to \mathcal{B}$.

Let us try to reinterpret the above result physically. Recall that $\mathcal{A}$ was the restriction of local operators in a gravity theory to a line, and $\mathcal{B}$ was the algebra of operators for a quantum mechanical system living along a line defect. We claim that the Koszul dual $\mathcal{A}^!$ should be thought of as the algebra of operators on a universal line defect in the gravitational theory. Indeed, note that there is a universal algebraic coupling $\alpha_{univ} \in \mathcal{A} \otimes \mathcal{A}^!$ corresponding to the identity map $\mathcal{A}^! \to \mathcal{A}^!$. This satisfies the following universal property: for any coupling $\alpha \in \mathcal{A} \otimes \mathcal{B}$,

$$\alpha = 1 \otimes \phi_\alpha(\alpha_{univ}).$$

Physically the choice of an augmentation corresponds to a vacuum in the gravitational theory. Note that a vacuum may not exist, and even if one exists there is no canonical way to construct one. Likewise augmentations may not always exist. The condition that an algebraic coupling be killed by the augmentation is precisely the condition that the defect we couple preserves the vacuum.

We can connect this story to how physicists speak about couplings. Let $\mathcal{O}(0) \in \mathcal{A}$ denote a local operator at $0 \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$ consider the translation $\tau_x \mathcal{O} = \mathcal{O}(x)$. Varying the point, we can think of this as defining a smooth family of local operators $\mathcal{O}(x) \in C^\infty(\mathbb{R}, \mathcal{A})$.

Suppose we are in a topological field theory, so each $\frac{\partial}{\partial x_i}$ acts homotopy trivially on $\mathcal{A}$. That is, there exist η^i acting on $\mathcal{A}$ such that

$$[Q_{\mathcal{A}}, \eta^i] = \frac{\partial}{\partial x_i}.$$

A procedure known as topological descent allows us to use this to define a 1-form operator. Consider

$$\mathcal{O}^{(1)}(x) = \sum_i (\eta^i \mathcal{O})(x) dx_i \in \Omega^1(\mathbb{R}^n, \mathcal{A}).$$

This satisfies the descent equation

$$d_{\mathrm{dR}} \mathcal{O}^{(0)}(x) = Q_{\mathcal{A}} \mathcal{O}^1(x).$$

We can iterate this to get an n-form local operator

$$\mathcal{O}^{(n)}(x) = \sum_i (\eta^i \mathcal{O}^{(n-1)})(x) dx_i.$$

The top form defines a lagrangian density

$$\int_{\mathbb{R}^n} \mathcal{O}^{(n)}(x)$$

and the descent equation implies that this expression is automatically $Q_{\mathcal{A}}$ closed.

The line defects that we discussed will only use the case where $n = 1$. Running descent on the coupling $\alpha \in \mathcal{A} \otimes \mathcal{B}$, we get a 1-form operator $\alpha^{(1)} \in \Omega^1(\mathbb{R}, \mathcal{A} \otimes \mathcal{B})$ and hence a lagrangian density $\int_{\mathbb{R}} \alpha^{(1)}$. The fact that the coupling satisfies a Maurer-Cartan equation in fact guarantees that this lagrangian density is BRST invariant to all orders in perturbation theory.

Example 2.12. Let us try and couple a system of N free bosons on a line to the theory in example **??**. The algebra of operators of the latter was found to be $\mathcal{A} = C^\bullet(\mathfrak{gl}_N) = \mathbb{C}[c_a]$ where $|c_a| = 1$ and the algebra of operators of bosonic quantum mechanics $\mathcal{B}$ is the Weyl algebra on N generators $\mathbb{C}[p^i, q_j]$ where $[p^i, q_j] = \delta^i_j$.

The defining $\mathfrak{gl}_N$ action gives us a Maurer-Cartan element

$$\alpha = \rho_i^{aj} c_a p^i q_j \in \mathfrak{gl}_N^* \otimes \mathcal{B} \subset \mathcal{A} \otimes \mathcal{B};$$

the Maurer-Cartan equation is equivalent to the dcondition that ρ in fact defines a representation.

Let us construct the corresponding lagrangian density. To run descent, we need a trivialization of the action of translations in the direction normal to the defect. Letting t denote a coordinate along the direction transverse to the line on which the bosonic quantum mechanics lives, the Cartan homotopy formula tells us that $\eta = \iota_{\partial_t}$ is a trivialization of infinitsesimal translations in the t direciton. When we run descent the induced action of η on observables has the effect of replacing the ghost with the gauge field; the result is

$$\alpha^{(1)} = \rho_j^{ai} A_a p^j q_i.$$

Integrating this recovers the familiar coupling

$$\int_{\mathbb{R}} (p, Aq).$$

2.4. **Baby Holography.** Let's start with Chern-Simons theory on $\mathbb{R} \times \mathbb{R}^2$ with a line defect along $\mathbb{R}$. Recalling the prescription from the first lecture, we wish to excise the defect and compactify on its link. Compactifying along the link of the line gives BF theory on the half space $\mathbb{R} \times \mathbb{R}_{\geq 0}$.

The fields of the theory are

$$A \in \Omega^\bullet(\mathbb{R} \times \mathbb{R}_{\geq 0}) \otimes \mathfrak{gl}_N[1]$$
$$B \in \Omega^\bullet(\mathbb{R} \times \mathbb{R}_{\geq 0}) \otimes \mathfrak{gl}_N$$

There are two boundary conditions that we can impose on either end of $\mathbb{R}_{>0}$

- At 0 we can impose a "Neumann" boundary condition by setting $B = 0$. Since B came from the fundamental class of the link of the defect, we can think of this boundary condition as filling in the location of the brane. The local operators here are the same as the local operators of the original gauge theory, $C^\bullet(\mathfrak{gl}_N)$.
- At ∞ we can impose a "Dirichlet" boundary condition where we set $A = 0$. One can show that the local operators on this brane is $U(\mathfrak{g})$.

Thus we arrive at a "baby" version of the relation between holography and Koszul duality. There is an equivalence

$$C^\bullet(\mathfrak{g})^! \cong U\mathfrak{g};$$

the former is the koszul dual of the local operators in Chern-Simons (the "gravitational" theory in our set up) restricted to a line, and the latter arises as some limit of the algebra of operators on a line defect.

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