A microscopic expansion for superconformal indices

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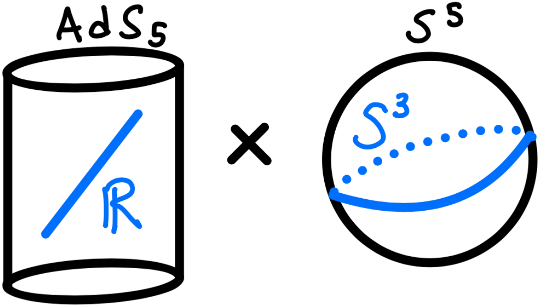
Today, I'd like to discuss a novel expansion of superconformal indices of $U(N)$ gauge theories at finite N.

When a holographic description is available, the formula expresses the index as a sum over stacks of certain branes in the dual string theory. These branes are so-called "giant graviton" branes wrapping $\mathbb{R} \times S^n \subset AdS \times M$ for M a compact manifold. They are particle-like in AdS but extend in the compact directions.



One of the central questions in quantum gravity is to understand and enumerate the microstates of black holes.

More precisely in the BPS sector, the question is the following: What is the Q-cohomology class of operators whose degeneracies give rise to those of BPS black holes at large energies/charges?

Holography, of course, makes this possible in principle. In recent years, there has been great progress in showing that the gauge theory index reproduces the entropy of BPS black holes in the gravity dual.[1]

However, the precise understanding of the operators, rather than their counting, has been missing.

[1] See works by Cabo-Bizet, Cassani, Martelli, Murthy, Choi, J. Kim, S. Kim, Nahmgoong, Benini, Milan, Honda, Arabi Ardehali, and more.

The index formula I discuss today conjecturally captures the $1/16$-BPS sector (up to signs) of $U(N)$ $\mathcal{N}=4$ super Yang-Mills in four dimensions at finite N, in terms of strings and branes in $AdS_5 \times S^5$.

In gauge theory, such strings and branes correspond to determinant operators and their modifications, dressed with usual operators of the multi-trace form.

At order 1 regime of charges, this sector is described by perturbative excitations of bulk strings. At order N^2 regime of charges, this sector is dominated by $1/16$-BPS AdS black holes. Our claim is that strings and giant graviton branes capture the degeneracy of states at all regimes of charges.

In this talk, I will consider the canonical example of the duality between IIB strings on $AdS_5 \times S^5$ and $U(N)$ $\mathcal{N} = 4$ super Yang-Mills, but similar expansion formulas apply to other holographic SCFTs such as the M2 worldvolume theory.

In particular, the Schur specialization of the $\mathcal{N} = 4$ SYM index gives the supersymmetric partition function of the $U(N)$-gauged beta-gamma system in twisted holography.

AdS/CFT and indices

Before we get there, let's review what holography says about partition functions and indices.

A way of stating AdS/CFT is as an equivalence of partition functions

$$\mathcal{Z}_{AdS \times M} = \mathcal{Z}_{CFT},$$

where the bulk metric and fields are supplemented with boundary conditions that specify the dual CFT.

The path integral of AdS quantum gravity, in the leading approximation at large N, can be computed as a sum over saddle geometries that asymptote to AdS,

$$\mathcal{Z}_{AdS} = \sum_{\text{geometries}} e^{-S}.$$

This sum namely includes pure AdS and many families of AdS black holes.

The statement of AdS/CFT descends to the (much more tractable) equivalence between the indices of gauge/string theories:

$$Z_{AdS \times M} = Z_{CFT}.$$

In gauge theory, the superconformal index

$$Z_{CFT} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS}}} \, (-1)^F e^{-\frac{1}{2}\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}$$

captures the BPS sector of the full Hilbert space, which is protected by superconformal symmetry. Sectors which are not BPS carry an equal number of bosons and fermions, which cancel due to $(-1)^F$. BPS states/operators are annihilated by the Poincaré/conformal supercharges $\mathcal{Q}, \mathcal{Q}^\dagger$, so $e^{-\frac{1}{2}\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}$ evaluates to 1.

It is helpful to "refine" the index by organizing the Hilbert space into representations that transform under the global symmetry charges $\mathcal{C}_i$:

$$Z_{CFT} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS}}} \, (-1)^F e^{-\frac{1}{2}\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} y_1^{\mathcal{C}_1} y_2^{\mathcal{C}_2} \cdots y_s^{\mathcal{C}_s},$$

for charges $\mathcal{C}_i$ that commute with $\mathcal{Q}, \mathcal{Q}^\dagger$. The fugacities y_i are related to chemical potentials as $y_i = e^{-\mu_i}$.

The coefficient of fugacities y_i in the index gives the degeneracy of BPS states with corresponding charge numbers, up to signs.

The index is topological; it is invariant under continuous deformations such as the change in couplings, size of manifold, or RG flow. Since we can continuously deform the CFT to a free theory and compute the index there, the index is much more tractable than a partition function.

Summing over geometries

Given that the CFT partition function $\mathcal{Z}_{CFT}$ is dual to a sum over geometries that asymptote to $AdS \times M$, one may wonder whether the index Z_{CFT} can be expressed directly as a sum over geometries.

Indeed, there has been much recent progress in finding black hole saddles for the index of $\mathcal{N} = 4$ SYM. In particular, $Z_{\mathcal{N}=4}$ can be expressed at large N as a sum over an infinite family of saddles as [2]

$$Z_{\mathcal{N}=4} = \sum_{m,n} e^{-N^2 S_{\text{eff}}(m,n;\tau)}$$

$$N^2 S_{\text{eff}}(0,1;\tau) = 0 \qquad \text{(pure AdS)}$$
$$N^2 S_{\text{eff}}(1,0;\tau) = F_{BH}(\tau) \qquad \text{(AdS BH).}$$

F_{BH} agrees with the free energy of a $\frac{1}{16}$-BPS black hole in AdS_5, computed with supersymmetric boundary conditions.

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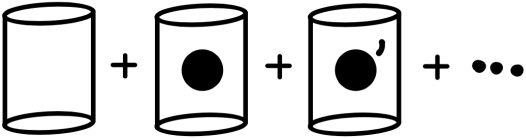
[2] Cabo-Bizet, Murthy '19; Cabo-Bizet, Cassani, Martelli, Murthy '20

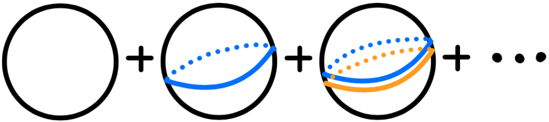
This formula is valid as an asymptotic expression at large N, and they capture the degeneracy of states at order $O(N^2)$ in the charges.

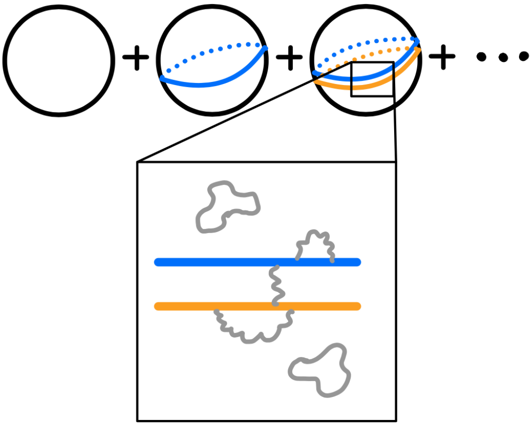


Summing over strings and branes

In this talk, I'd like to present a different expansion of the $U(N)$ gauge theory index, where objects in the sum have a natural interpretation in the dual string theory as D3 giant graviton branes in $AdS_5 \times S^5$.

$$Z_N(x; y_i) = Z_\infty(x; y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y_i)\right].$$





$$Z_N(x; y_i) = Z_\infty(x; y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y_i)\right].$$

The prefactor $Z_\infty(x; y_i)$ captures the perturbative spectrum at infinite N, which can be matched with the supergraviton spectrum in the bulk. The sum represents stacks of $0, 1, 2, \cdots$ numbers of $D3$ giant graviton branes wrapping $\mathbb{R} \times S^3 \subset AdS_5 \times S^5$.

The "brane indices" $\hat{Z}_k(x; y_i)$ are related to gauge theory indices $Z_N(x; y_i)$ by an involution acting on the fugacities. $\{x, y_i\}$ are a collection of fugacities, but I've singled out the fugacity x corresponding to a bosonic adjoint operator.

We emphasize that the expansion is a statement purely in the gauge theory, though the objects involved admit very natural interpretations in the bulk.

Let's compare our expansion to the sum over geometries. One expects to see the following in the spectrum of IIB string theory in $AdS_5 \times S^5$:

- At charges of $O(1)$, perturbative excitations of $AdS_5 \times S^5$ corresponding to supergravitons/strings
- At charges of $O(N)$, non-perturbative excitations such as D-branes
- At charges of $O(N^2)$, fully backreacted, highly excited geometries such as BPS black holes

So the giant graviton expansion is quite different from the sum over on-shell geometries, since branes contribute to the index at order $O(N)$ rather than $O(N^2)$ in the charges. Therefore, the sum over branes is not a standard sum over saddles.

Before discussing the ingredients in more detail, let us mention some strange and surprising properties of the expansion.

$$Z_N(x; y_i) = Z_\infty(x; y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y_i)\right].$$

(1) From the spectrum, one may expect that the giant graviton expansion will break down at charges of $O(N^2)$ due to extra contributions from saddle geometries. However, explicit computations show that such contributions are not necessary. The above expansion holds much beyond $O(N^2)$ of charges.

$$Z_N(x; y_i) = Z_\infty(x; y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN}\hat{Z}_k(x; y_i)\right].$$

This suggests that the sum over branes somehow already accounts for the degeneracy of states coming from the sum over saddle geometries at charges of $O(N^2)$. It does so in a "microscopic" manner, interpolating between the $O(1)$ charge regime of perturbative excitations and $O(N^2)$ regime of geometries.

$$Z_N(x; y_i) = Z_\infty(x; y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y_i)\right].$$

A subtlety here is that the picture in terms of giant gravitons is likely appropriate for small t'Hooft coupling λ, while saddle geometries are valid at large λ. However, the index is agnostic about λ.

$$Z_N(x;y_i) = Z_\infty(x;y_i)\left[1+\sum_{k=1}^{\infty} x^{kN}\hat{Z}_k(x;y_i)\right].$$

(2) The expansion also holds at low (even for zero and negative) N, where the dual string theory is highly nonperturbative, i.e. when the string coupling g_s is large. In this regime, supergravity breaks down so a black hole solution is not well-defined.

Despite this, the validity of the formula at low N suggests that some remnant of the picture in terms of strings and branes persists even in the large g_s regime.

$$Z_N(x; y_i) = Z_\infty(x; y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y_i)\right].$$

(3) The validity for negative N suggests a combinatorial origin of the formula. In fact, the expansion also applies to many index-like quantities. This perhaps may be justified, as large N t'Hooft combinatorics tells us that any $U(N)$ gauge theory is dual to some string theory, even if that string theory does not admit a weakly-curved gravity description.

$$Z_N(x; y_i) = Z_\infty(x; y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y_i)\right].$$

(4) I've made a choice of the fugacity x, which corresponds to the bosonic adjoint scalar X. In $\mathcal{N} = 4$, there are two other equivalent choices y, z corresponding to the fields Y, Z. The choice of x, y, z corresponds to the three independent ways to embed $S^3 \subset S^5$. It turns out that considering just one orientation is sufficient to reproduce the correct degeneracies, when we look at the series expansion in x at fixed orders of y and z.

$$Z_N(x;y_i) = Z_\infty(x;y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN}\hat{Z}_k(x;y_i)\right].$$

A naive counting of all orientations will give redundant results, but a careful counting of all three orientations gives a result that agrees with our proposal. See work by Y. Imamura.[3]

[3] Imamura '21

Let's now derive the strings and brane terms in the expansion. To do so requires some knowledge of how to compute superconformal indices.

Computing the index

As mentioned before, the index can be computed in the free theory where the couplings are turned off.

In a free theory, the computation becomes a combinatorial problem of counting the ways in which different "letters" L_i, corresponding to BPS fields and their derivatives, can be combined into gauge-invariant operators, which are polynomials of L_i.

A standard trick to solve this problem is to first count all the single letters L_i, which one collects it into a function $f(y_i)$ called the single letter index. Then one applies an operation

$$PE[f(y_i)] = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f(y_i^n)\right),$$

called the plethystic exponential, which automatically counts the ways in which L_i are combined into polynomials.

For example, a $\mathcal{N} = 1$ chiral multiplet (and its conjugate pair) consists of two BPS fields: a bosonic scalar X and a fermion $\bar{\psi}_X$. The single letters are BPS derivatives of these fields and take the form

$$\partial_1^n \partial_2^m X, \qquad \partial_1^n \partial_2^m \bar{\psi}_X.$$

In a certain basis of charges, we can assign them the following fugacities:

$$p^n q^m x, \qquad -p^{n+1} q^{m+1} x^{-1}.$$

Summing all such letters, we get the single letter index

$$f(x,p,q) = \sum_{n,m} (x - x^{-1}pq)p^m q^n = \frac{x - x^{-1}pq}{(1-p)(1-q)}.$$

The generating function that counts the polynomials of the letters is

$$Z(x,p,q) = PE[f(x,p,q)] = \prod_{n,m} \frac{1 - p^{n+1}q^{m+1}x^{-1}}{1 - p^n q^m x},$$

which one may recognize as the elliptic gamma function $\Gamma(x;p,q)$. Notice that the numerator came from fermions and the denominator came from bosons. This is the index of a chiral multiplet in a $\mathcal{N} = 1$ theory.

In a $U(N)$ gauge theory, there can be fugacities μ_a associated with gauge charges, as well as fugacities y_i for global symmetry charges. In such cases, the single letter index takes the form

$$f(y_i)\ \mathrm{Tr}U\ \mathrm{Tr}U^{-1} = f(y_i) \sum_a \mu_a \sum_b \mu_b^{-1}$$

We project onto gauge-invariant polynomials of BPS letters by treating μ_a as $U(N)$-eigenvalues and integrating over $U(N)$:

$$Z_N(y_i) = \frac{1}{N!} \int \frac{d\mu}{2\pi i \mu_a} \prod_{a\neq b} (1-\mu_a\mu_b^{-1})\ \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f(y_i^n) \sum_a \mu_a^n \sum_b \mu_b^{-n} \right).$$

the extra factor in the measure is the Vandermonde determinant for $U(N)$. The integral extracts the singlet sector of the integrand under the gauge group $U(N)$.

$\mathcal{O}(1)$: Strings

We now derive the prefactor from closed strings in the formula

$$Z_N(x;y_i) = Z_\infty(x;y_i)\left[1+\sum_{k=1}^{\infty} x^{kN}\hat{Z}_k(x;y_i)\right].$$

This contribution is given by the index evaluated at $N=\infty$.

The $U(N)$ index can be computed exactly in the limit where N is taken to infinity. In this limit, the N-dependence drops out. It can be shown that the gauge integral becomes gaussian and evaluates to

$$Z_\infty(y_i) = \prod_{n=1}^{\infty}\frac{1}{1-f(y_i^n)}$$

When considering brane contributions shortly, it will also be important that the infinite N formula is readily generalized to theories with (anti-)fundamental matter with indices $v(y_i), \bar{v}(y_i)$:

$$Z_\infty(y_i) = \prod_{n=1}^{\infty} \frac{1}{1 - f(y_i^n)} PE\Big[\frac{v\bar{v}}{1-f}\Big].$$

To ground ourselves, let's take the very simple example of a $U(N)$ theory with a single adjoint scalar X without any derivatives. Here, gauge-invariant operators look like $\mathrm{Tr} X^n$, so we are counting independent polynomials of $\mathrm{Tr} X^n$, up to trace relations. Let's take $N = \infty$, so that there are no trace relations.

These polynomials are generated by

$$
\begin{aligned}
&\text{Level 0:} \quad 1 \\
&\text{Level 1:} \quad \mathrm{Tr}\, X, \\
&\text{Level 2:} \quad \mathrm{Tr}\, X^2, \quad (\mathrm{Tr}\, X)^2 \\
&\text{Level 3:} \quad \mathrm{Tr}\, X^3, \quad (\mathrm{Tr}\, X^2)(\mathrm{Tr}\, X), \quad (\mathrm{Tr}\, X)^3 \\
&\text{Level 4:} \quad \mathrm{Tr}\, X^4, \quad (\mathrm{Tr}\, X^3)(\mathrm{Tr}\, X), \quad (\mathrm{Tr}\, X^2)(\mathrm{Tr}\, X^2), \\
&\qquad\qquad\;\; (\mathrm{Tr}\, X^2)(\mathrm{Tr}\, X)^2, \quad (\mathrm{Tr}\, X)^4
\end{aligned}
$$

and so on.

Let's now check the operator counting against the gaussian formula. The single letter index here is just $f = x$. At infinite N, the generating function is

$$Z_\infty(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots ,$$

which matches the counting.

The $U(N)$ $\mathcal{N}=4$ SYM index is a variation on the same theme. The index can be expressed in terms of fugacities x, y, z of three adjoint scalars X, Y, Z and fugacities p, q for two BPS derivatives. There are also conjugate fermions $\bar{\psi}_{X,Y,Z}$ from the chiral multiplets. The fugacities obey the constraint $xyz = pq$.

The single letter index can be collected into the nice form

$$f = 1 - \frac{(1-x)(1-y)(1-z)}{(1-p)(1-q)}$$

with the infinite N index

$$Z_\infty(x, y, z, p, q) = \prod_{n=1}^{\infty} \frac{(1-p^n)(1-q^n)}{(1-x^n)(1-y^n)(1-z^n)}.$$

The power expansion coefficients count the independent gauge-invariant polynomials of $1/16$-BPS $\mathcal{N}=4$ SYM operators, up to signs and with no trace relations.

It is a standard result that infinite N index of $\mathcal{N} = 4$ SYM agrees with the supergraviton spectrum in $AdS_5 \times S^5$. We have derived the prefactor in the brane expansion formula that accounts for $O(1)$ perturbative excitations.

$\mathcal{O}(N)$: Giant graviton branes

Now let's derive the brane terms that show up on the right hand side of the formula

$$Z_N(x; y_i) = Z_\infty(x; y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y_i)\right].$$

It will be necessary to assume that N is large for the derivation. The justification for the formula at finite N comes from direct computational checks.

In $\mathcal{N} = 4$ SYM, determinant operators have dimension N and are known to be dual to $D3$ giant graviton branes that wrap $\mathbb{R} \times S^3 \subset AdS_5 \times S^5$. Finite modifications of the determinants correspond to open string excitations of these $D3$s.

So the index of determinant operators along with their modifications in gauge theory will account for giant gravitons in the holographic dual.

We start with a single determinant

$$\det X \propto \epsilon^{i_1 i_2 \cdots i_N} \epsilon_{j_1 j_2 \cdots j_N} X^{j_1}_{i_1} X^{j_2}_{i_2} \cdots X^{j_N}_{i_N}$$

of fugacity x^N.

The determinant can be modified by replacing a finite number of above X's by strings of letters in the theory. For example, we can replace an X with $L_1 L_2 L_1$ to get

$$\epsilon^{i_1 i_2 \cdots i_N} \epsilon_{j_1 j_2 \cdots j_N} (L_1 L_2 L_1)^{j_1}_{i_1} X^{j_2}_{i_2} \cdots X^{j_N}_{i_N}.$$

There are some redundancies with counting such modifications, though. An example is a replacement of the form $X \to XW$, where antisymmetry allows one to write $\mathrm{Tr} W \det X$. We will interpret these redundancies as closed strings and include them back later. For now, let's focus on the nonredundant part that corresponds to open strings.

A helpful reformulation of the problem is to write the determinant as an integral over auxiliary (anti-)fundamental fermions

$$\int d\bar{\psi} d\psi \; e^{\bar{\psi} X \psi}.$$

Introducing (anti-)fundamental fermions adds "boundaries" to the t'Hooft ribbon diagrams, which helps understand why determinants should correspond to D-brane insertions.

In terms of the fermion integral, determinant modifications correspond to operator insertions

$$\int d\bar{\psi} d\psi \ e^{\bar{\psi} X \psi} (\bar{\psi} W_1 \psi)(\bar{\psi} W_2 \psi) \cdots$$

where an open string excitation looks like $\bar{\psi} L_1 \cdots L_s \psi$.

The redundancy mentioned above becomes a Ward identity for the fermions

$$\int d\bar{\psi} d\psi \ e^{\bar{\psi} X \psi} (\bar{\psi} X W \psi)(\bar{\psi} W_1 \psi) \cdots = \int d\bar{\psi} d\psi \ e^{\bar{\psi} X \psi} \left(\frac{d}{d\psi} (W \psi) \right) (\bar{\psi} W_1 \psi) \cdots .$$

We can implement the Ward identities by introducing bosonic antifields $u, \bar{u}$, as well as a BRST differential δ acting as

$$\delta X = \delta\psi = \delta\bar{\psi} = 0$$

$$\delta u = X\psi$$

$$\delta\bar{u} = \bar{\psi}X.$$

For proper counting, we will need to posit that δ is an extra part of the cohomological supercharge $\mathcal{Q}$ that acts nontrivially only on the antifields $u, \bar{u}$.

We assign ghost numbers -1 to $u, \bar{u}$, $+1$ to δ, and 0 to other fields. We are interested in operators in the BRST cohomology with ghost number 0. The action of δ were written so that redundancies due to replacements $X \to XW$ or $X \to WX$ become δ-exact.

The auxiliary fundamental and anti-fundamental letters are counted by the single letters $v = (x - 1)\lambda$ and $\bar{v} = (1 - x^{-1})\lambda^{-1}$. λ denotes a fugacity for an extra $U(1)$ symmetry which only acts on these auxiliary variables. It will drop out of calculations now but will be useful soon.

One problem with this approach is that there is cohomology in non-zero ghost number. The operator $\bar{\psi} X \psi$ can come from either $\delta(\bar{\psi} u)$ and $\delta(\bar{u}\psi)$, so the combination $\bar{\psi} u + \bar{u}\psi$ will be δ-closed but not exact. It gives a fermionic zeromode with ghost number -1 and trivial fugacity. As this operator is the only problematic one, we will simply remove the zeromode by hand in our counting.

Let's now look back at the large N gaussian index formula with (anti-)fundamentals

$$Z_\infty(y_i) = \prod_{n=1}^{\infty} \frac{1}{1-f(y_i^n)} PE\left[\frac{v\bar{v}}{1-f}\right],$$

which I mentioned would be useful. The infinite prefactor is the redundant closed string sector, which we'll ignore for now.

The large N formula suggests that the "effective" single letter index governing the determinant fluctuations is

$$\tilde{f} = 1 + \frac{v\bar{v}}{1-f} = 1 - \frac{(1-x)(1-x^{-1})}{1-f}$$

where the extra factor of 1 cancels the zeromode.

Therefore, modifications of a single determinant, with the redundant sector stripped off, are counted by the tilded index

$$\tilde{Z}_1 = PE[\tilde{f}].$$

Perhaps a more enlightening rearrangement of the relation between f and $\tilde{f}$ is

$$(1-f)(1-\tilde{f}) = (1-x)(1-\tilde{x}),$$

with $\tilde{x} = x^{-1}$. This relates the gauge theory index and the determinant modification index via the involution $\sim$.

Let's apply this relation to the single adjoint matrix case with $f = x$. We get

$$\tilde{f} = \tilde{x} = x^{-1}.$$

This makes sense. Here, the only nontrivial operator is X, so any modification of the determinant would correspond to replacing X by the identity I. This would take away a single power of fugacity x, thus the inverse.

For $\mathcal{N} = 4$ SYM, we get

$$\tilde{f} = 1 - \frac{(1-\tilde{x})(1-p)(1-q)}{(1-y)(1-z)},$$

with $\tilde{x} = x^{-1}$. So the determinant modification index is related to the gauge theory index only by the exchange of fugacities

$$x \leftrightarrow \tilde{x}, \quad p \leftrightarrow y, \quad q \leftrightarrow z.$$

The determinant index should then be interpreted as a $U(1)$ gauge theory index on the worldvolume of a single $D3$ giant graviton, where the angular momentum and charge fugacities are swapped. x is mapped to its inverse, because determinant modifications remove X's.

It is straightforward to consider the modifications of k determinants using the fermion description with k flavors of fermions.

$$(\det X)^k = \int d\bar{\psi} d\psi \ e^{\bar{\psi}^{\alpha} X \psi_{\alpha}}$$

$$\int d\bar{\psi} d\psi \ e^{\bar{\psi}^{\alpha} X \psi_{\alpha}} (\bar{\psi}^{\beta} W_1 \psi_{\gamma})(\bar{\psi}^{\delta} W_2 \psi_{\epsilon})$$

Insertions with different fermion indices now correspond to open strings stretched between different pairs of k coincident giant graviton branes.

The only difference from the previous case is that there is an emergent $U(k)$ gauge symmetry on the giant graviton worldvolumes that must be imposed. For proper counting, we should also subtract k^2 fermion zeromodes by hand.

Modifications of k determinants are described by the index

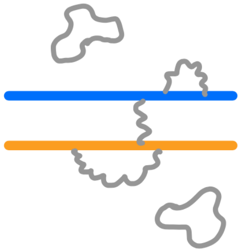
$$\tilde{Z}_k(\tilde{x}; y_i) = \frac{1}{N!} \int \frac{d\lambda}{2\pi i \lambda_a} \prod_{a \neq b} (1 - \lambda_a \lambda_b^{-1}) \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \tilde{f}(y_i^n) \sum_a \lambda_a^n \sum_b \lambda_b^{-n} \right)$$

with $\tilde{f}$ defined in the same way as in the $U(1)$ case. λ_a are fugacities for an extra $U(N)$ symmetry that only act on the auxiliary variables. It dropped out for $U(1)$, but they become gauge fugacities for $U(N)$.

Let's put back in the closed string sector and the bare fugacity x^{kN} for k determinants. The index of k giant gravitons and their fluctuations is

$$x^{kN} Z_\infty(x; y_i) \tilde{Z}_k(\tilde{x}; y_i).$$

x^{kN} is the fugacity of the bare determinant $(\det X)^k$, $Z_\infty(x; y_i)$ are background excitations of $AdS_5 \times S^5$, and $\tilde{Z}_k(\tilde{x}; y_i)$ are open string excitations on k coincident giant gravitons.



Summing over strings and branes

The expansion formula is simply a sum of the above ingredients over all numbers of $D3$ giant gravitons:

$$Z_N(x; y_i) = Z_\infty(x; y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y_i)\right].$$

The fugacities $\tilde{x}$ in $\tilde{Z}_k$ have been analytically continued outside the unit disk in the index $\hat{Z}_k$, so that $|\tilde{x}^{-1}| = |x| < 1$ in $\hat{Z}_k$. The analytic continuation is tractable in many explicit examples, because the infinite series in x can often be resummed to a rational function.

We derived the individual ingredients, i.e. strings and branes, in the expansion formula but have not justified analytically that the sum of these ingredients gives the finite N superconformal index. However, explicit computations provide good evidence that the index formula is valid at any order of the charges and for any integer N.

If the formula holds true, we are led to the conjecture that determinant operators and their modifications, multiplied by usual operators of the multi-trace form, exhaust the Q-cohomology at finite N.

Strictly speaking, this conjecture applies at zero 't Hooft coupling in usual gauge theory. However, this conjecture should hold exactly in twisted holography.

In the example of the single adjoint matrix, it is not difficult to prove that the formula holds exactly:

$$Z_N(x) := \frac{1}{\prod_{n=0}^{N}(1-x^n)} = \frac{1}{\prod_{n=0}^{\infty}(1-x^n)} \sum_{k=0}^{\infty} \frac{x^{kN}}{\prod_{n=0}^{k}(1-x^{-n})}.$$

For $\mathcal{N}=4$ SYM and its specializations, the formula can be verified as a power series x, at any given powers of other fugacities y_i. See our paper for many checks; direct matching with black hole degeneracies in progress.

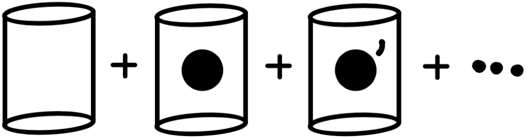
That said, it would be interesting if there is a localization argument that can be used to prove the formula.

Concluding remarks

It seems now that there are two different expressions for the index of $\mathcal{N} = 4$ SYM in terms of bulk objects.

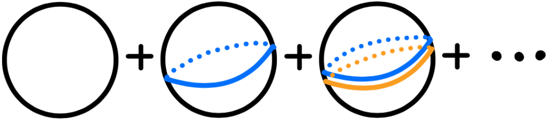
One is a large N saddle expansion in terms of black holes, which makes the macroscopic geometry manifest.

$$Z_{\mathcal{N}=4} = \sum_{m,n} e^{-N^2 S_{\text{eff}}(m,n;\tau)}$$



The other is a finite N combinatorial expansion in terms of strings and branes, which makes the bulk microstates of the BPS sector manifest.

$$Z_N(x; y_i) = Z_\infty(x; y_i)\left[1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y_i)\right].$$



Now that we have the exact microscopic ingredients, it would be interesting to understand how the strings and branes reorganize themselves into various geometries.

Thank you