Twisted Holography of M2 and M5 branes

(Partly based on work with Davide Gaiotto and with Yehao Zhou)

Focus on protected subsector of AdS/CFT by twisting both sides.

Field Theory: ① Pick a nilpotent supercharge Q ($Q^2 = 0$)

② Focus on Q-cohomology ($\leftrightarrow$ associative algebra of operators)

Question: What's the analogue for SUGRA side?

Recall: Supersymmetry is local (gauge) symmetry in SUGRA.

↑ parametrizes

Ψ (Killing spinor) $\xrightarrow[\text{Theory}]{\text{Field}}$ ε (SUSY parameter)

To study gauge theory systematically, we introduce ghosts (boson) for the gauge symmetry (= local supersymmetry)

Definition Twisted Supergravity is SUGRA with particular components of ghosts for local SUSY nonzero. [Costello]

In practice, better to recall field theory case.

Example: 4d $\mathcal{N}=1$ SUSY with a chiral multiplet $\Phi = (\phi, \psi_\alpha)$, $\bar{\Phi} = (\bar{\phi}, \bar{\psi}_{\dot\alpha})$. Note SUSY TR is generated by Q_α, $\bar{Q}_{\dot\alpha}$. We holomorphically twist the theory by picking $Q = Q_-$, and focus on Q-cohomology

↕ set $\varepsilon_+ = 1$, others zero

$\delta\bar{\phi} = \bar{\varepsilon}\bar{\psi}$, $\delta\phi = \varepsilon_+\psi_- - \varepsilon_-\psi_+$

$\rightsquigarrow \delta_Q\bar{\phi} = 0$, $\delta_Q\phi = \psi_-$ $\Rightarrow$ $\bar{\phi} \in$ Q-cohomology

Similarly, to twist SUGRA, we only keep particular components of Ψ. (In rigid limit, $\Psi \to \varepsilon$).

Twisted holography : Duality b/w twisted QFT & SUGRA.

Advantages : ① Everything becomes algebraic (can compare spectrum)

② With some constraints (Ω-background), gravity side simplifies $\rightsquigarrow$ 5d top-hol Chern-Simons

↳ soon explain.

Preliminary : ①	Topological	Holomorphic twist
Lorentz ⊕ R → Lorentz' (↑ twist) d-dim QFT	Q's → [Q (scalar), $\mathbf{Q}$ (1-form)] such that $\{Q, \mathbf{Q}\} = P_\mu$, $\mu = 1, \cdots, d$	Q's → [Q (scalar), $\mathbf{Q}$ (1-form)] such that $\{Q, \mathbf{Q}\} = P_{\bar{i}}$, $i = 1, \cdots, d/2$
Passing to Q-coh	$\mathcal{O}(x_\mu) \to \mathcal{O}$	$\Rightarrow \mathcal{O}(z_i, \bar{z}_i) \to \mathcal{O}(z_i)$

② Ω_ε-deformation on twisted QFT

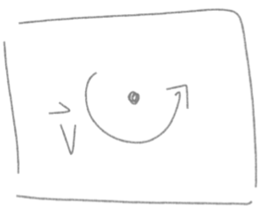
Condition : $U(1)$ isometry in the spacetime (e.g. $\mathbb{R}^2$) (generated by V_ε)

Twisted QFT = Q-cohomology equipped with $\mathbf{Q}$ (1-form)

With Ω_ε, = Q_ε-cohomology " " "

$$Q_\varepsilon = Q + i_V \mathbf{Q}, \quad Q_\varepsilon^2 = \mathcal{L}_V \neq 0$$

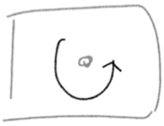
$\Rightarrow$ Q_ε-coh $\subset$ Q-coh with



Operators localize at the fixed point of V.

Similarly in twisted SUGRA, we may find Ψ for

① topological-holomorphic background

② $\Psi \xrightarrow{\Omega_\varepsilon} \Psi_\varepsilon$ such that $\Psi_\varepsilon^2 = \mathcal{L}_V$. Again, 

Will focus on twisted holography for M-theory.

Gravity Background: Ψ_ε, g, C with

g : $(G_2)_7 \times (\text{hyperKähler})_4 = (\underbrace{\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\varepsilon_2} \times \mathbb{C}_{\varepsilon_3}}_{\circlearrowright TN_k} \times \mathbb{R}) \times (\mathbb{C}_z \times \mathbb{C}_w)$

C : $V^b \wedge \omega_{0,2}$

Ψ_ε : M_7 (top), M_4 (hol) & $\Omega_{\varepsilon_1}, \Omega_{\varepsilon_2}, \Omega_{\varepsilon_3}$ on $\mathbb{C}^3 \subset M_7$

$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \ : \ CY_3)$

Field theory: worldvolume theory on

N_1 M2 on $\mathbb{C}_{\varepsilon_i} \times \mathbb{R}_t$

N_2 M5 on $\mathbb{C}_{\varepsilon_i} \times \mathbb{C}_{\varepsilon_j} \times \mathbb{C}_z$

Useful to go to type IIA frame by reducing $S^1 \subset TN_k$

Gravity background $\rightsquigarrow$ IIA SUGRA ⊕ k D6 branes & B-field ($\rightarrow$ Non-commutative background on $\mathbb{C}_z \times \mathbb{C}_w$) ($C \nearrow$ B-field)

[Costello]: 6 top (A) ⊕ 4 hol (B) with B-field = 10 top (A) $\rightsquigarrow$ trivial closed strings

$\rightsquigarrow$ Only get k D6 branes w/ B-field.

D6 branes $\equiv$ 7d SYM on $\underbrace{\mathbb{C}_{\varepsilon_1} \times \mathbb{R}_t}_{top} \times \underbrace{\mathbb{C}_z \times \mathbb{C}_w}_{hol}$

By [Costello, Yagi], 7d SYM on $\mathbb{C}_{\varepsilon_1} \underset{\text{localization}}{\equiv}$ 5d top-hol Chern-Simons theory.

$S_{5d\,CS} = \frac{1}{\varepsilon_1}\int dz\,dw \left(A\,dA + \frac{2}{3} A \underset{\varepsilon_2}{*} A \underset{\varepsilon_2}{*} A \right)$ on $\mathbb{R}_t \times \mathbb{C}_z \times \mathbb{C}_w$

($A \to A_t dt + A_{\bar z} d\bar z + A_{\bar w} d\bar w$)

$\left(f \underset{\varepsilon_2}{*} g = fg + \varepsilon_2 (\partial_z f \partial_w g - \partial_w f \partial_z g) + (\varepsilon_2^2/2)(\cdots) + \cdots \right)$

└ Moyal product induced by non-commutative background.

[Costello] : $S_{5d\,CS}$ is renormalizable & gauge invariant.

Gauge symmetry : $U_{\varepsilon_1}(\mathcal{O}_{\mathbb{C}^2_{\varepsilon_2}} \otimes gl_K)$ → Major difference

Algebra of "observables" → Call it $Obs_{5d}^{\varepsilon_1,\varepsilon_2}$

$\Lambda(z,w) \sim \sum_{m,n} z^m w^n (T^a_b)\, \partial_z^m \partial_w^n \Lambda$, $[z,w] = \varepsilon_2$

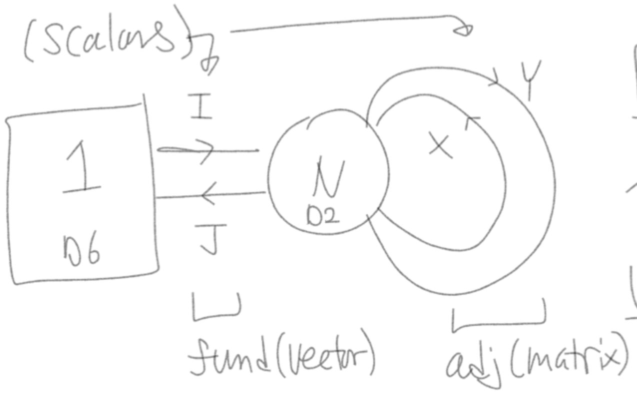
($z^m w^n \in \mathcal{O}_{\mathbb{C}^2_{\varepsilon_2}}$, $T^a_b \in gl_K$)

Field theory I on N M2 branes (on $\mathbb{R}_t \times \mathbb{C}_{\varepsilon_1}$ (Both top))

In type IIa, they are N D2 branes.

Due to (background) D6, worldvolume theory on D2 : 3d $\mathcal{N}=4$ (→ UV gauge theory of ADHM)

$G = U(N)$, with fundamental hyper & adjoint hyper



Recall Topological twist on $\mathbb{R}_t \times \mathbb{C}_{\varepsilon_1}$

⇝ choose Q_{RW} (Q_{tRW} possible)

Known Q_{RW} - cohomology = Higgs branch (X, Y, I, J)

Observables Higgs branch chiral ring = gauge invariant words of X, Y, I, J modulo F-term relation

(e.g. $I X^m Y^n J$ or $\mathrm{Tr}\, X^m Y^n$)

F-term relation = $[X,Y] + JI = \varepsilon_2 1_{N\times N}$

$\equiv \{ t_{m,n} = I X^m Y^n J \}$ → Call it $A_{\varepsilon_1,\varepsilon_2}$

Ω_{ε_1}-background quantizes the Higgs chiral ring into an algebra $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$

$$\begin{pmatrix} \{X,Y\}_{PB} \to [X^a{}_b, Y^c{}_d] = \varepsilon_1 \delta^a_d \delta^c_b \\ \{I,J\}_{PB} \to [I_a, J^b] = \varepsilon_1 \delta^b_a \end{pmatrix}$$

↳ a,b,c,d: $U(N)$ gauge index

(1d) Lagrangian: $\int_{\mathbb{R}_t} X dY + I dJ + \varepsilon_2 \int \mathrm{Tr}\, A \quad (3d \to 1d)$

Example commutator: (take K D6 for simplicity) a,b,c,d: $U(K)$ flavor index

$$[(t_{1,0})^a{}_b, (t_{0,1})^c{}_d] = \varepsilon_1 (t_{0,0})^a{}_d (t_{0,0})^c{}_b \quad \text{[Will see later]}$$

Note $t_{0,0} = N\varepsilon_1$ (in large N limit, treat it as a central element of $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$)

[Costello] $\mathrm{Obs}^{\varepsilon_1,\varepsilon_2}_{5d} \leftrightarrow \mathcal{A}_{\varepsilon_1,\varepsilon_2}$ (Koszul dual)

Dictionary: $z^m w^n (T^a) \in \mathcal{O}(\mathbb{C}^2_{\varepsilon_2}) \otimes gl_K \longleftrightarrow I X^m Y^n J$

Underlying principle: Koszul duality [exchanges ghost # = 0 operators ($\mathcal{A}_{\varepsilon_1,\varepsilon_2}$: physical) & ghost # > 0 operators ($\mathrm{Obs}^{\varepsilon_1,\varepsilon_2}_{5d}$)]

It induces a unique coupling b/w Obs_{5d} & $\mathcal{A}^{1d}_{\varepsilon_1,\varepsilon_2}$ such that for $c \in \mathrm{obs}^{5d}$, $t \in \mathcal{A}^{1d}$, $x = c \otimes t \in \mathrm{Obs}_{5d} \otimes \mathcal{A}^{1d}$ satisfies

Maurer–Cartan equation $dX + \frac{1}{2}[X,X] = 0$

↳ ensures BRST invariance of coupling

Explicitly, the coupling $= \int_{\mathbb{R}_t} (\partial_z^m \partial_w^n A_t)\, t_{m,n} = S^{\text{couple}}_{1d}$

unique QM-consistent coupling.

Concrete way to see →

S^{CS}_{5d} is already BRST-invariant. What about $S^{CS}_{5d} + S^{couple}_{1d}$?

__Strategy__ Imposing BRST-invariance of 5d/1d system will uniquely fixes $\mathcal{A}$.

__How?__ Compute Feynman diagram of 5d/1d interaction. and impose $\sum_i \delta_{BRST}(\text{Feynman diagram})_i$

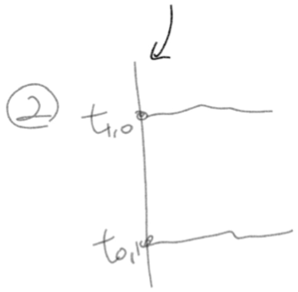
e.g.

$\propto \quad t_{0,0}\, t_{0,0}\, \partial_z A\, \partial_w A$

$\downarrow \delta_{BRST} A = c$

$\varepsilon_1\, t_{0,0}\, t_{0,0}\, \partial_z A\, \partial_w c$

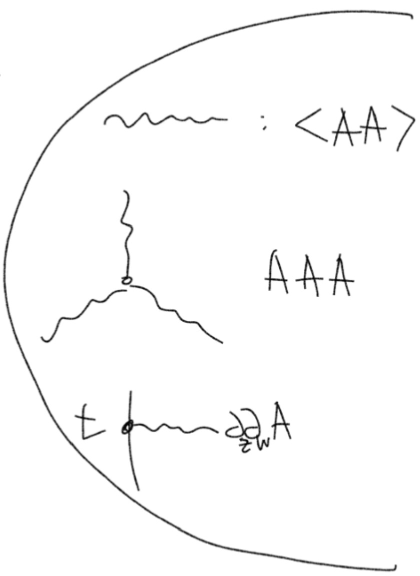
(ε_1: loop counting parameter)



$\propto \quad t_{1,0}\, t_{0,1}\, \partial_z A\, \partial_w A$

$\downarrow \delta_{BRST} A = c$

$t_{1,0}\, t_{0,1}\, \partial_z A\, \partial_w c$



Impose ① = ② ⟹ $t_{1,0}\, t_{0,1} = \varepsilon_1\, t_{0,0}\, t_{0,0}$

Exactly $\mathcal{A}_{\varepsilon_1, \varepsilon_2}$!

__Note__ ① Large N is necessary to match the algebra

② Roughly, twisted holography of M2 branes is a subsector of [ABJM] ($AdS_2 \times S^3 \subset AdS_4 \times S^7$)

③ Can equally use Q_{tRW} & 3d $\mathcal{N}=4$ Coulomb branch algebra to arrive at the same algebra (1-shifted affine gl(1) Yangian) → useful to compare with M5-brane example

Field theory II on N M5 branes on $\mathbb{C}_z \times \mathbb{C}_{\varepsilon_i} \times \mathbb{C}_{\varepsilon_j}$.
(hol) (top)

Worldvolume theory: 6d A_N type (2,0) theory.

* 6d ^{A_N}(2,0) w/ double Ω $\xrightarrow{[AGT]}$ W_N algebra on $\mathbb{C}_z$ [Yagi; Beem et al, Bobev et al]

In large N limit, $W_N \to W_\infty$ algebra.

* W_∞ algebra $\cong$ Affine $gl(1)$ Yangian [Prochazka]

The above argument goes through and in this case,

$$\mathrm{Obs}_{5d\ CS} \cong U_{\varepsilon_1}(\mathcal{O}(\mathbb{C}_z \times_{\varepsilon_2} \mathbb{C}^\times_w) \otimes gl_1) \cong W_\infty \text{ algebra } [\text{Costello}]$$

- $\mathbb{C}^\times_w$ ← due to M5-branes
- gl_1 ← 1 D6-brane
- W_∞ algebra ← $\oint \mathrm{Tr}_{gl(N)} \bar{\psi} A z^m \partial_z^n \psi$ ($\bar\psi$, ψ: D4-D6 strings)

<u>Question</u>: where's $\varepsilon_1, \varepsilon_2$ in W_∞? ($\varepsilon_3 = -\varepsilon_1 - \varepsilon_2$) [M2: quantum($\varepsilon_1$), Non-Commutativity($\varepsilon_2$)]

<u>Known</u> ① Hidden triality in W_∞ [Gaberdiel, Gopakumar]

② Affine $gl(1)$ Yangian has $\varepsilon_1, \varepsilon_2$ parameters. ($\varepsilon_3 = -\varepsilon_1 - \varepsilon_2$)

Koszul duality ⇝ coupling b/w 5d & 2d system via

Unique BRST-inv coupling: $S^{2d}_{\text{coupling}} = \int_{\mathbb{C}_z} (\partial_z^{m-1} A)\, W^{(m)}$, $m \in \mathbb{Z}_+$

$W^{(m)}$ ↳ Spin m current. (e.g. $m=2 \to T$)

<u>More generally</u>

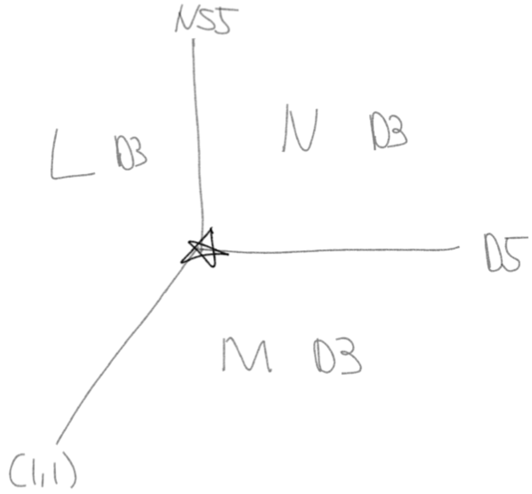
		Hol	Top		
L	$M5_1$ on	$\mathbb{C}_z$ ×	$\mathbb{C}_{\varepsilon_1}$ ×	$\mathbb{C}_{\varepsilon_2}$	
M	$M5_2$ on	$\mathbb{C}_z$ ×	$\mathbb{C}_{\varepsilon_1}$		× $\mathbb{C}_{\varepsilon_3}$
N	$M5_3$ on	$\mathbb{C}_z$		× $\mathbb{C}_{\varepsilon_2}$	× $\mathbb{C}_{\varepsilon_3}$

(<u>Recall</u> Hol: $\mathbb{C}_z \times \mathbb{C}_w$
Top: $\mathbb{R}_t \times \mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\varepsilon_2} \times \mathbb{C}_{\varepsilon_3}$)

Maps to Corner VOA configuration in <u>type IIB</u> via [Gaiotto, Rapcak]

$M/T^2 \cong IIB/S^1$.

↓



At the corner, ☆ we have

$Y_{L,M,N} \subset W_\infty$ → NS5 ∩ D5 ∩ D3

↓

a truncation of W_∞

Note ① 5d top-hol CS is the boundary condition of D5-brane on NS5 brane, living on ☆ (boundary condition → NS5 ∩ D5)

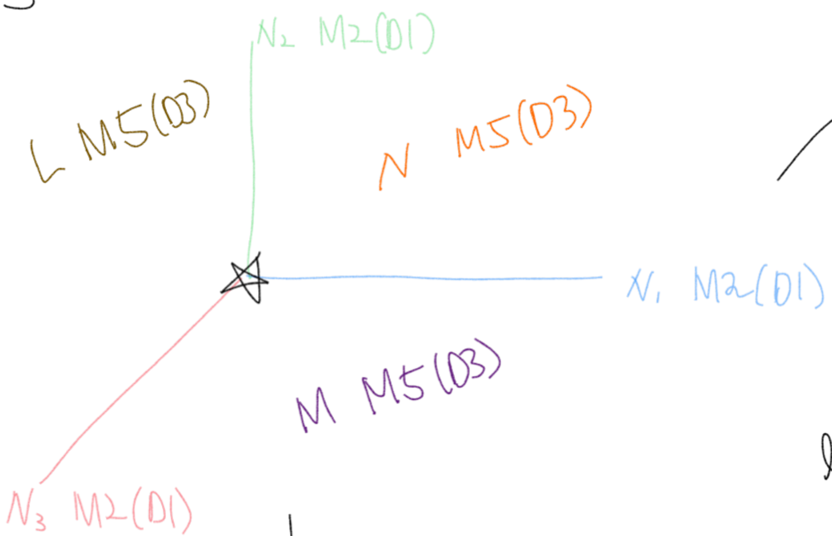
② N_1 M2 branes on $\mathbb{R}_t \times \mathbb{C}_{\epsilon_1}$

N_2 M2 branes on $\mathbb{R}_t \times \mathbb{C}_{\epsilon_2}$ map to

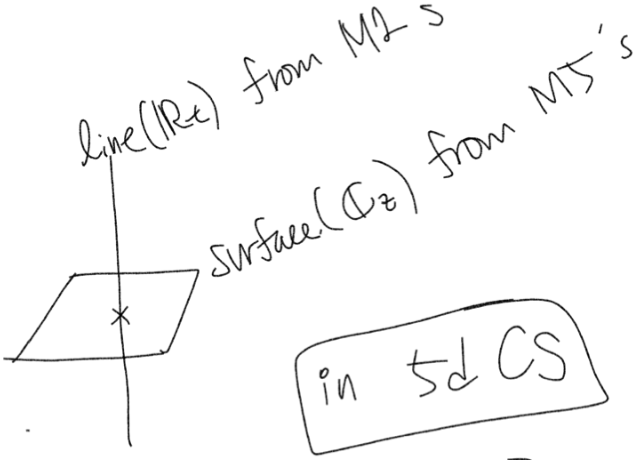
N_3 M2 branes on $\mathbb{R}_t \times \mathbb{C}_{\epsilon_3}$

each edge of the web-diagram. { (1,0)-string, (0,1)-string, (1,1)-string }

Putting all together, in type IIB frame, we have



Look into the corner ☆



↓

Most general algebras { $\mathcal{A}_{N_1,N_2,N_3}$ (M2), $\mathcal{W}_{L,M,N}$ (M5) }

in 5d CS

[Junction of line & Surface]

Want to construct $\begin{Bmatrix} \mathcal{A}_{N_1,N_2,N_3} \\ \mathcal{W}_{L,M,N} \end{Bmatrix}$ → Fusion of defects & Coproduct

[Gaiotto, Rapcak]

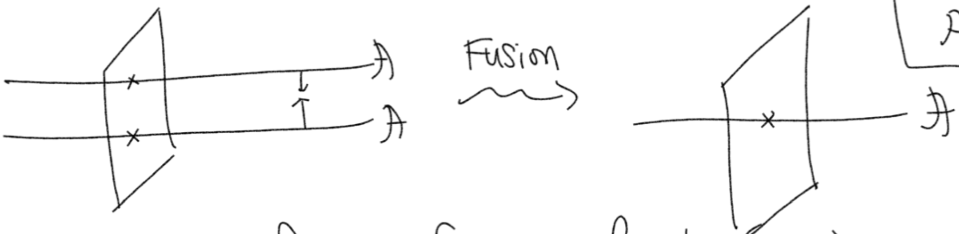
$\mathcal{A}_{N_1 N_2 N_3} \to \mathcal{A}_{N_1 00} \otimes \mathcal{A}_{0 N_2 0} \otimes \mathcal{A}_{00 N_3}$

$\mathcal{W}_{LMN} \to \mathcal{W}_{L00} \otimes \mathcal{W}_{0M0} \otimes \mathcal{W}_{00N}$

using free field realization of $\mathcal{A}$ and $\mathcal{W}_{\infty}$

Three basic operations :

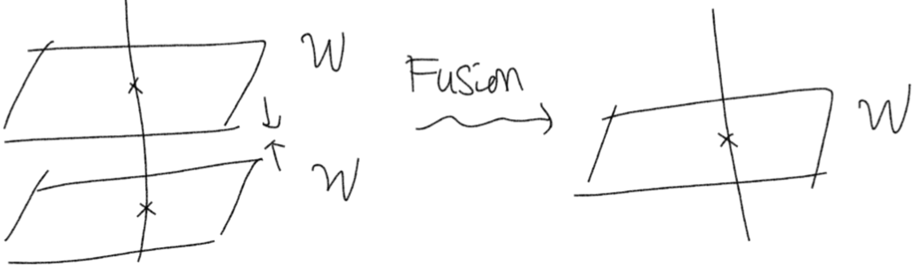
① Fusion of line defects (M2)



induces $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$\boxed{\Delta_{\mathcal{A},\mathcal{A}}} : t \longmapsto t \otimes t$

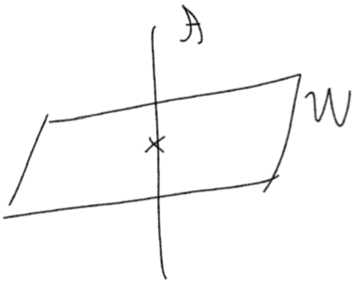
② Fusion of surface defects (M5)



induces $\mathcal{W} \to \mathcal{W} \otimes \mathcal{W}$

$\boxed{\Delta_{\mathcal{W},\mathcal{W}}} : W \longmapsto W \otimes W$

③ BRST invariance of the junction (M2 - M5)

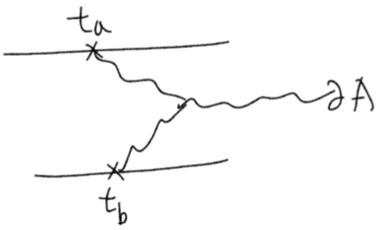


induces $\mathcal{A} \to \mathcal{A} \otimes \mathcal{W}$

$\boxed{\Delta_{\mathcal{A},\mathcal{W}}} : t \longmapsto t \otimes W$

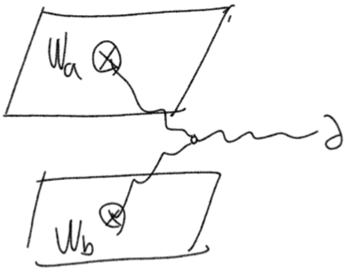
Holographic derivation : Compute Feynman diagrams [Oh, Zhou]

① Homogeneous fusion : OPE between lines & Surfaces.

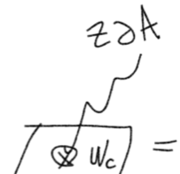


$= t_a t_b \int_{\mathbb{R}_t} \partial A =$

$= t_c \int_{\mathbb{R}_t} \partial A \Rightarrow \boxed{\Delta_{AA} : t \to t \otimes t}$



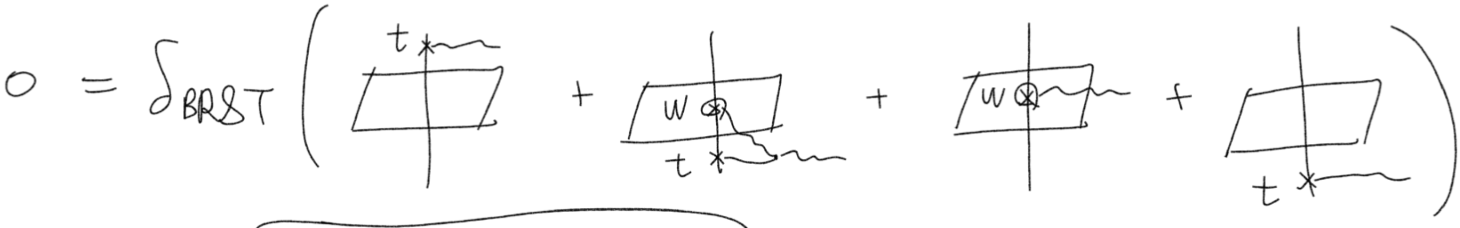
$= W_a W_b \int_{\mathbb{C}_z} z \partial A =$



$= W_c \int_{\mathbb{C}_z} z \partial A$

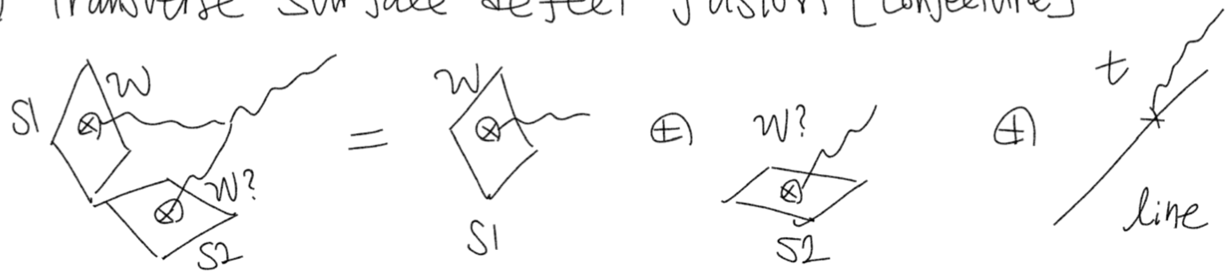
$\Rightarrow \boxed{\Delta_{W,W} : W \to W \otimes W}$

② Heterotic fusion (Impose BRST invariance)

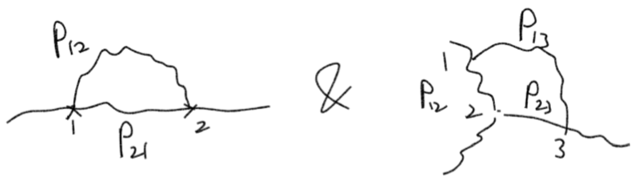


$\Rightarrow$ $\boxed{\Delta_{A,W} : A \to A \otimes \mathcal{W}}$

③ Transverse surface defect fusion [Conjecture]



④ All 1-loop exact due to



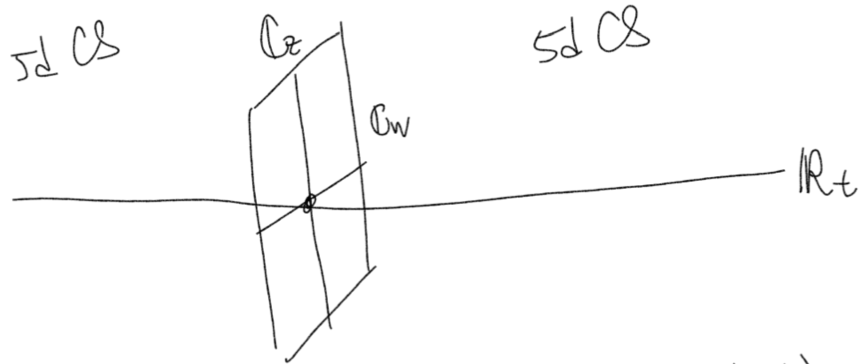
$\hookrightarrow O(\sigma_3)$ → $\epsilon_1 \epsilon_2 \epsilon_3$ loop counting)

$P_{12} \wedge P_{21} = 0$ $\qquad$ $P_{12} \wedge P_{23} \wedge P_{31} = 0$

$\Downarrow$

$O(\sigma_3^{n \geq 2}) = 0$.

* 4d domain wall & higher VOA on $\mathbb{C}_z \times \mathbb{C}_w$ in 5d CS on $\mathbb{R}_t \times \mathbb{C}_z \times \mathbb{C}_w$. [Oh, Zhou]



Domain wall (higher VOA)

$TN_K \otimes TN_N$

* Other generalization: change $\mathbb{R} \times \mathbb{C}_{\epsilon_1} \times TN_K \to G_2$ manifold

Related 5d CS? What is the algebra? [Del Zotto, Oh, Zhou]

[Oh, Zhou] $\qquad$ Roughly, two copies of $\mathcal{U}(\mathrm{Diff}\,\mathbb{C} \otimes gl_K)$

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