Introduction to Twisted String Theory and Twisted Holography

Holography says (for example)

Type IIB String Theory on

$AdS^5 \times S^5 \cong \mathcal{N}=4$ SYM, $G=U(N)$

$N \to \infty$ on S^4

Lorentzian
$\mathbb{H}^5$

Mathematicians: ???

Even for physicists, hard to formulate rigorously:

- $\mathcal{N}=4$ SYM must be defined non-perturbatively (??)
- $AdS^5 \times S^5$ is challenging for string theory, RR background

Goal of this seminar:

Look at twists (aka, supersymmetric subsectors) of both sides, and examine the duality there

$N=4$ SYM ⇝ familiar mathematical things:

Langlands, vertex algebras, geometric rep. theory

String theory ⇝ somewhat familiar things: topological strings in various dimensions.

<u>What is twisting?</u>

SUSY QFT has an action of super Poincaré algebra:

$$(S \oplus \mathbb{R}^n) \rtimes so(n)$$

Some spin rep

Also have $G_R \subseteq \operatorname{End} S$ commutant of $Spin(n)$

Twisting, Step 1:

Choose a homomorphism $\rho : Spin(n) \to G_R$ to change action of $Spin(n)$ on fields.

My perspective: This does nothing (locally).

Not that important.

Twisting, step 2

Choose $Q \in S$, $[Q, Q] = 0$

and add Q to the differential of everything in the QFT:

$$d \rightsquigarrow d + Q$$

Better Choice of Q gives an action of Abelian superalgebra

$\Pi\mathbb{C}$ on the theory

$\Pi\mathbb{C}$ derived invariants are a module for $\mathcal{O}(B\Pi\mathbb{C}) = \mathbb{C}[[t]]$

If we have a $\mathbb{C}^\times$ action where Q has weight 1 then we have a Rees family - twisted theory lives over generic point.

2nd part of twisting: radical simplification of the theory.

Variant: $Q^2 =$ a rotation

Can perform equivariant cohomology construction:

Q-cohomology on S^1 fixed points

"Ω-background":

write $\mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}^2_{\varepsilon_2}$ .. if $Q^2 = \varepsilon_1 \partial_{\theta_1} + \varepsilon_2 \partial_{\theta_2}$

Examples of Twists

$N=4$ SYM on $\mathbb{R}^2 \times \mathbb{C}$

top' hol.

or $\mathbb{R}^2 \times \Sigma$ is "classical" geometric Langlands (Arinkin)

$$\mathrm{Coh}(\mathrm{Higgs}_G(\Sigma)) \leftrightarrow \mathrm{Coh}(\mathrm{Higgs}_{\check{G}}(\Sigma))$$

On $\mathbb{R}^2_\varepsilon \times \mathbb{C}$

$N=4$ SYM $\rightsquigarrow$ a VOA:

BRST reduction of CDOs on $\mathfrak{g}$ by G, adjoint action

$=$ CDOs on $\mathfrak{g}/G$ adjoint quotient stack.

M2 brane on

$$\mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}^2_{\varepsilon_2} \times \underbrace{\mathbb{R}^2_{\varepsilon_3} \times \mathbb{R}}_{M2} \times \mathbb{C}^2$$

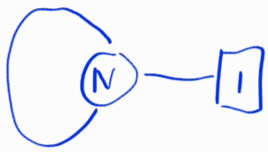
gives ADHM quantum mechanics. Algebra of operators is quantum Hamiltonian reduction of $T^*(\mathfrak{gl}_N \oplus \mathbb{C}^N)$

by GL_N

= Spherical DAHA

= Deformation quantization of $\mathrm{Hilb}^N(\mathbb{C}^2)$

= Coulomb branch algebra for

ADHM quiver 

Twisting Supergravity (a sketch)

In supergravity, super-symmetries are gauged: fields are a stack

$$\frac{(\text{metric, etc. etc.})}{(\text{A big supergroup})}$$

The big supergroup is very roughly maps from

$$\mathbb{R}^n \longrightarrow Spin(n) \ltimes (\mathbb{R}^n \oplus \Pi S)$$

So, fields lives over maps

$$\mathbb{R}^n \longrightarrow \underbrace{B(Spin(n) \ltimes (\mathbb{R}^n \oplus \Pi S))}_{}$$

Analog of "moduli of vacua"

$$B(\mathfrak{g}) = MC(\mathfrak{g}) = \left\{ \text{Maurer-Cartan elements} \,/\, \text{Gauge} \right\}$$

$$MC(\mathbb{R}^n \oplus \Pi S) = \text{"nilpotence variety"}$$

$$\{ Q \in \Pi S, \ [Q, Q] = 0 \}$$

Conclude, super-gravity contains a field which looks like a map

$\mathbb{R}^n \to$ Nilpotence Variety $/ Spin(n)$

("bosonic ghost" - ghost for gauged fermionic symmetries)

Twisted supergravity: we work in a background where this field takes some non-zero constant value (or tends to such at ∞)

This is a vacuum for ordinary supergravity.

Examples

II B string on $\mathbb{R}^2_\varepsilon \times \mathbb{R}^2_{-\varepsilon} \times X$

(X a CY3) $\rightsquigarrow$ B model on X

II A string on $\mathbb{R}^2_\varepsilon \times \mathbb{R}^2_{-\varepsilon} \times X$

$\rightsquigarrow$ A model on X

This was known since the early days (early 90s) in different language

$\mathbb{R}^2_\varepsilon \times \mathbb{R}^2_{-\varepsilon} \rightsquigarrow$ "graviphoton" turned on, + SUSY localization

Dedushenko - Witten: more modern treatment.

11d supergravity on

$$\mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}^2_{\varepsilon_2} \times \mathbb{R}^2_{\varepsilon_3} \times \mathbb{R} \times \mathbb{C}^2$$

$\rightsquigarrow$ non-commutative Chern-Simons theory:

$$A \in \Omega^1(\mathbb{R} \times \mathbb{C}^2) \text{ mod } dz_1, dz_2$$

$$\frac{1}{\varepsilon_3} \int dz_1 dz_2 \left(\frac{1}{2} A * dA + \frac{1}{3} A * A * A \right)$$

$f * g$ = Moyal product

$$fg + \delta \varepsilon^{ij} \partial_{z_i} f \partial_{z_j} g + \ldots$$

$$\frac{1}{\varepsilon_3} \int dz_1 dz_2 \left(\frac{1}{2} A dA + \delta \varepsilon^{ij} A \partial_{z_i} A \partial_{z_j} A + \ldots \right)$$

Examples of Holography

M2 branes:

Spherical DAHA for gl_N as $N \to \infty$

$\longleftrightarrow$ Non commutative CS on $\mathbb{R} \times \mathbb{C}^2$

D3 branes:

CDOs on gl_N / GL_N

$\longleftrightarrow$ B model on $\mathbb{C}^3$ (actually geometry gets modified)

How to relate gauge theory to gravity? Full story: a little involved (Koszul duality / boundary operators)

Will first describe a simpler computation

Gauge Theory $\longrightarrow$ A Lie algebra $\mathfrak{g}_{gauge}$

Gravity $\longrightarrow$ A Lie algebra $\mathfrak{g}_{grav}$

We will check they are isomorphic

$$\mathfrak{g}_{grav} \cong \mathfrak{g}_{gauge}$$

In physical AdS/CFT this is very simple:

$\mathfrak{g}_{grav}$ = Gauge transformations of $AdS_5 \times S^5$ gravity fixing the metric

= Isometries (+ fermionic symmetries)

$$= \underbrace{SO(6)}_{} \times \underbrace{SO(4,2)}_{\text{Conformal symmetries of } S^4}$$

R symmetry of $N=4$: 6 scalars in fundamental rep, etc.

So, $\mathfrak{g}_{grav}$ is obvious (in fact all) symmetries of $\mathcal{N}=4$ SYM

Next example:

$\mathbb{R}\times\mathbb{C}^2$ NC Chern-Simons theory

$\longleftrightarrow$ Large N spherical DAHA

What are gauge symmetries of NC Chern-Simons?

$$A \in \Omega^1(\mathbb{R}\times\mathbb{C}^2) \text{ mod } dz_1, dz_2$$

$$A = A_t + A_{\bar{z}_1} + A_{\bar{z}_2}$$

Gauge transformations

$$A \to A + \bar{\partial}\chi + d_t\chi + [\chi, A]$$

Moyal commutator

$$\varepsilon_{ij}\,\partial_{z_i}\chi\,\partial_{z_j}A + \cdots$$

Gauge transformations that preserve

$A=0$ are χ with $d\chi = 0$ mod dz_1, dz_2

So $\partial_t\chi = 0$ $\quad \partial_{\bar{z}_i}\chi = 0$ $\quad \chi \in \mathcal{O}(\mathbb{C}^2)$

with bracket for Moyal product

Conclude: gauge transformations are

$$\mathrm{Diff}(\mathbb{C})$$

Gauge theory side: how do we see $\mathrm{Diff}(\mathbb{C})$ from ADHM quiver?

$$I_i \in \mathbb{C}^N \qquad J^i \in (\mathbb{C}^N)^*$$

$$X^i_j\ ,\ Y^i_j \in \mathfrak{gl}_N$$

Moment map

$$[X,Y] + IJ = c \leftarrow \text{generic}$$

$N \gg 0$ Want to describe functions on symplectic quotient (moment map relation + GL_N invariance)

Lemma $\quad I X^r Y^s J$

generate the algebra of functions on symplectic quotient, and are algebraically independent if $r+s<N$

Proof The fact that they generate is a consequence of the moment map relation and classical invariant theory.

Independence requires a more detailed argument. $\square$.

$N \to \infty$: functions on symplectic quotient are

$$S^*(\mathcal{O}(\mathbb{C}^2))$$

$$I X^r Y^s J \longleftrightarrow z_1^r z_2^s$$

Quantum version is obtained by quantum Hamiltonian reduction $\rightsquigarrow$ some flat deformation A of $S^* \mathcal{O}(\mathbb{C}^2)$

At the quantum level we find a quantum deformation of $U \underbrace{\mathrm{Diff}\, \mathbb{C}^2}$

what we found in gauge theory

Sketch: $X^i_j \quad Y^i_j \quad I_i \quad J^i$ live in a Weyl algebra:

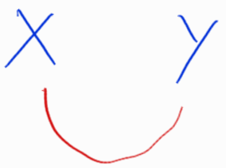
$$[X^i_j, Y^k_l] = \hbar \delta^i_l \delta^k_j$$
$$[I_i, J^j] = \hbar \delta_i^j$$

Compute commutators of the expressions

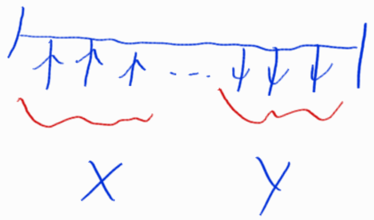
$$I X^r Y^s J = I_{i_0} X^{i_0}_{i_1} X^{i_1}_{i_2} \cdots Y^{i_{r+s}}_{i_{r+s+1}} J^{i_{r+s+1}}$$

$$X^i_j = \hbar \frac{\partial}{\partial Y^j_i} \qquad I_i = \hbar \frac{\partial}{\partial J^i}$$

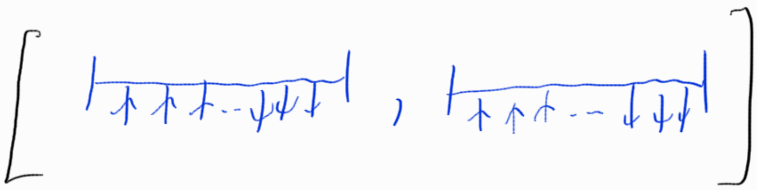
Diagramatic notation



means in the commutator $\hbar \frac{\partial}{\partial Y} = X$ eats Y

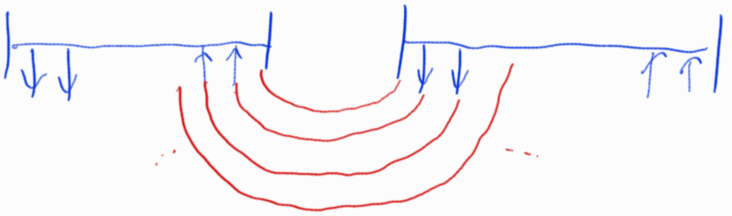


The commutator is computed by



various contractions

Dominant term (planar limit) is



gives a Lie bracket on "single string" operators
$I X^r Y^s J$

Lemma This Lie algebra is $\mathrm{Diff}(\mathbb{C})$

In this limit,

Large N quantum Hamiltonian reduction
$= \; U\mathrm{Diff}(\mathbb{C})$

where $1 \in \mathrm{Diff}(\mathbb{C})$ becomes N

(moment map relation: $JI + [X, Y] = 1$
take trace, $IJ = N$)

Questions 1) How does this relate to physicists $AdS_m \times S^n$ picture?

2) How do we access the full algebra of large N DAHA including more complicated terms

$[I X^r Y^s J, I X^m Y^n J] =$ linear

+ quadratic and higher

expressions in the $I X^k Y^l J$ generators

Answers 1) We'll see

2) Using Feynman diagrams + Koszul duality

Where is AdS

Usual physics algorithm:

1) Brane on $\mathbb{R}^d \subseteq \mathbb{R}^n$
Solve EOM for gravitational fields in the presence of the brane

2) Metric has singularities on $\mathbb{R}^d$; remove this locus leaving

$$\left(\mathbb{R}^n \setminus \mathbb{R}^d, g\right)$$ black brane geometry

3) Zoom in near $\mathbb{R}^d$ "near horizon limit" gives $AdS_{d+1} \times S^{n-d-1}$

Not needed in the twisted setting, 1) and 2) suffice

For $\mathbb{R} \subseteq \mathbb{R} \times \mathbb{C}^2$

$$\begin{array}{ccc} \mathbb{R} & & \mathbb{R}\times\mathbb{C}^2 \\ \swarrow & & \searrow \\ m_2 & & \text{NC Chern-Simons} \end{array}$$

A gauge field of 5d CS. To find field sourced by the brane we solve EOM in the presence of the source term

$$N\int_{\mathbb{R}} A \quad + \int_{\mathbb{R}\times\mathbb{C}^2} dz_1\, dz_2\, A\, dA$$

Solution A is Bochner-Martinelli kernel

$$A = N\frac{\bar z_1 d\bar z_2 - \bar z_2 d\bar z_1}{\|z\|^4}$$

So, $\mathbb{R}\times(\mathbb{C}^2\setminus 0)$ with background is analog of $AdS_2\times S^3$

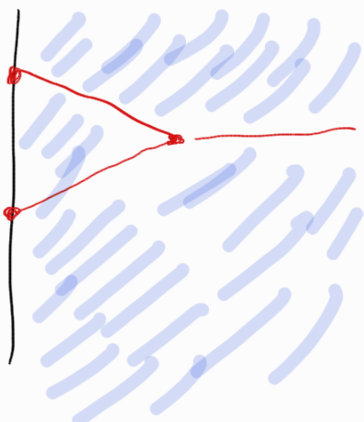
Effective $2d$ theory turns out to be Poisson σ-model for $\mathrm{Diff}(\mathbb{C})^*$ with Poisson tensor that from the Lie bracket on $\mathrm{Diff}(\mathbb{C})$

$r = \|z\|$

$\mathbb{R}\times\mathbb{R}_{>0}$ Poisson σ-model, boundary condition at $r=\infty$ is Neumann (as used by Kontsevich)

Algebra of boundary operators, at tree level (Kontsevich) is

$U(\mathrm{Diff}(\mathbb{C}))$



This is the physics picture:

Large N CFT algebra

$=$ boundary algebra for theory on AdS_2 (including all KK modes)

In our context, back reaction

$$A = N \frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{\|z\|^4}$$

does very little - only identifies

$1 \in Diff(\mathbb{C})$

with N.

(Alternative: take Poisson σ-model for $[Diff(\mathbb{C})/1]^*$ backreaction field A adds a new term to the Poisson σ-model action)

2 approaches:

1) Take the mode corresponding to backreaction as dynamical.

Then, boundary algebra will have a central element corresponding to N

2) Take the mode corresponding to the back-reaction to be non-dynamical. Then N is a parameter (more common in physics treatments).

Before turning to quantum aspects, let's look at the other basic example:

$$\begin{tikzcd} \mathbb{C} \arrow[d, dashed] & \subseteq & \mathbb{C}^3 \arrow[d, dashed] \\ \text{CDOs on } \mathfrak{gl}_N/GL_N & & \text{B-model} \end{tikzcd}$$

Fields in the B-model include Beltrami differential

$$\mu \in \Omega^{0,1}(\mathbb{C}^3; T\mathbb{C}^3) \cong \Omega^{2,1}(\mathbb{C}^3)$$

couples to brane by $N\int_{\mathbb{C}} \partial^{-1}\mu$

Solutions to the EOM in presence of source term is

$$\mu = N \frac{(\bar{w}_1 d\bar{w}_2 - \bar{w}_2 d\bar{w}_1)}{\|w\|^4} \partial_z$$

Coordinates z, w_1, w_2 brane at $w_i = 0$

Lemma $\mathbb{C} \times (\mathbb{C}^2 \setminus 0)$ deformed by this Beltrami differential is

$$SL_2\mathbb{C}$$

Proof: Holomorphic functions are w_1, w_2 and

$$v_1 = z w_1 - \frac{N \bar{w}_2}{\|w\|^2}$$

$$v_2 = z w_2 + \frac{N \bar{w}_1}{\|w\|^2}$$

$$v_2 w_1 - v_1 w_2 = N \qquad \square.$$

Then, as for 5d theory, can ask to match

Boundary operators of B-model

$\leftrightarrow$ large N limit of CDOs

on $\mathfrak{gl}_N / GL_N$

Gauge symmetries

$\leftrightarrow$ Symmetries.

Gauge symmetries of B-model on $SL_2(\mathbb{C})$ are the Lie algebra

$X \in \mathrm{Vect}_0(SL_2(\mathbb{C}))$ $(D_\Omega X = 0)$

$f, g \in \Pi\, \mathcal{O}(SL_2(\mathbb{C}))$

$[X, -]$ = Lie derivative

$[f, g]$ = vector field so

$[f,g] \vee \Omega = df \wedge dg$

Call this $\mathfrak{g}_{grav}$

For gauge theory side, consider

A_N = Mode algebra of of CDOs on $\mathfrak{gl}_N / GL_N$

<u>Theorem</u> (C, Davide Gaiotto)

As $N \to \infty$ there is an embedding

$U\mathfrak{g}_{grav} \hookrightarrow A_\infty$

$\mathfrak{g}_{grav}$ = single trace modes that preserve the vacuum at 0 and ∞

$\Rightarrow$ these modes preserve all correlation functions "global symmetries"

In this case, backreaction is essential.
Can not treat field soured by defect as dynamical because brane is coupled by

$$\int_{\mathbb{C}} \partial^{-1} \mu$$

Koszul Duality

Back to 5d story.

$$\mathbb{R} \times (\mathbb{C}^2 \setminus 0) = \mathbb{R} \times S^3 \times \mathbb{R}_{>0}$$

Reduction

$$\rightsquigarrow \mathbb{R} \times \mathbb{R}_{>0}$$

5d gauge theory $\Longrightarrow$ Poisson σ model for $\mathrm{Diff}(\mathbb{C})$

$=$ BF theory on $\mathbb{R} \times \mathbb{R}_{>0}$ for $\mathrm{Diff}(\mathbb{C})$

$A \in \Omega^1(\mathbb{R} \times \mathbb{R}_{>0}, \mathrm{Diff}\,\mathbb{C})$ a connection

$B \in \Omega^0(\mathbb{R} \times \mathbb{R}_{>0}, (\mathrm{Diff}\,\mathbb{C})^*)$

$$\int \langle B, F(A) \rangle$$

At $r = \infty$ we ask that $A = 0$ and boundary operators (functions of B) are (classically) $U\,\mathrm{Diff}\,\mathbb{C}$

Gauge theory side: $U\,\mathrm{Diff}\,\mathbb{C}$ deforms to a quantized universal enveloping algebra

How to see this on gravity side?

Problem 2d analysis fails.

Solution Koszul duality.

General principle: In any 2d TFT if B_1, B_2 are boundary conditions such that $\mathrm{Hom}(B_1, B_2) = \mathbb{C}$ ("transverse") then $\mathrm{End}(B_1)$ and $\mathrm{End}(B_2)$ are Koszul dual.