Addendum to Jihwan's Talk :

More on $\mathrm{Obs}^{\varepsilon_1}(M2)$

Recall the setting :

$$\text{Twisted } \mathcal{M}\text{-theory on } \overbrace{\mathcal{M}_7^{G_2}}^{A\text{-twist}} \times \overbrace{\mathcal{M}_4^{HK}}^{B\text{-twist}}$$

$$\varepsilon_1+\varepsilon_2+\varepsilon_3=0$$

	$\mathbb{C}_{\varepsilon_1} \times$	$TN_k^{\varepsilon_2\varepsilon_3}$	$\times$	$\mathbb{R}_t \times$	$\mathbb{C}_z \times$	$\mathbb{C}_w$
$N_1\to\infty$ M2	X			X		
$N_2\to\infty$ M5		X	X		X	
II A on	$\mathbb{C}_{\varepsilon_1} \times$	$\mathbb{R}_{\varepsilon_2}^{x}$	$\mathbb{C}_{\varepsilon_3} \times$	$\mathbb{R}_t \times$	$\mathbb{C}_z \times$	$\mathbb{C}_w$
$N_1\to\infty$ D2	X			X		
$N_2\to\infty$ D1		X	X		X	
K D6	X			X	X	X

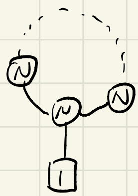
Reduce $S^1_{\varepsilon_2}$

Ω_{ε_1}-deform

5d hCS on $\mathbb{R}_t \times \mathbb{C}^2_{z,w}$, gauge group GL_k

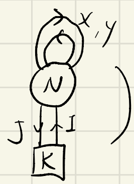
In this talk, we will focus on the algebra of observables on M2 brane, notation : $\mathrm{Obs}^{\varepsilon_1}(M2)$

<u>Recall</u>. In Jihwan's talk, we see that $\mathrm{Obs}^{\varepsilon_1}(M2)$ is the Coulomb branch algebra of 3d $\mathcal{N}=4$ quiver gauge theory of



with K gauge nodes

Or equivalently, it's the Higgs branch

$$\mathrm{Obs}^{\varepsilon_1}(M2) \simeq \mathcal{M}_H^{\varepsilon_1}\Big($$


$$\Big)$$

"ADHM QM"

$$[x,y]+IJ=0$$

<u>Remark</u> ① After turning on ε_2, it gives rise to a mass deformation of $\mathcal{M}_C$, $\Leftrightarrow$ FI parameter in $\mathcal{M}_H$,

$$[X,Y]+IJ=\varepsilon_2\cdot \mathrm{Id}$$

② $\mathcal{M}_H^{\varepsilon_1,\varepsilon_2}$ is a quantization of $\mathcal{M}_H^{\varepsilon_2}$

④ It seems more convenient working with generators and relations of $\mathcal{M}_H^{\varepsilon_1,\varepsilon_2}$, if we want to compare it with $\mathrm{Obs}^{\varepsilon_1}(5d\ CS)$, so we work with Higgs branch instead of Coulomb branch.

Quantum Moment Map & Quantum Hamiltonian Reduction

Definition: Let $\mathfrak{g}$ be Lie alg$/\mathbb{C}$, A be associative alg$/\mathbb{C}$ w./ a $\mathfrak{g}$-action, i.e. $\phi: \mathfrak{g} \to \mathrm{Der}(A)$

A quantum moment map for $(A, \mathfrak{g}, \phi)$ is a Lie alg. homomorphism $\mu: \mathfrak{g} \to A$ s.t.

$$[\mu(a), b] = \underbrace{\phi(a) \cdot b}_{\text{action}} \qquad a \in \mathfrak{g}, \quad b \in A$$

Lemma-Definition: Suppose that $(A, \mathfrak{g}, \phi)$ has q. moment map μ,

Let J be Left ideal of A generated by $\mu(\mathfrak{g})$, then $J^{\mathfrak{g}} = J \cap A^{\mathfrak{g}}$ is a two-sided ideal of $A^{\mathfrak{g}}$

proof: let $x = \sum b_i \mu(a_i) \in J^{\mathfrak{g}}$, and $y \in A^{\mathfrak{g}}$,

then $x \cdot y = \sum b_i \mu(a_i) \cdot y = \sum b_i y \cdot \mu(a_i) + \sum b_i [\mu(a_i), y]$

$$= \sum b_i \cdot y \cdot \mu(a_i) + \underbrace{\sum b_i \phi(a_i) \cdot y}_{= 0 \text{ since } y \in A^{\mathfrak{g}}}$$

$$\in J^{\mathfrak{g}}$$

We call $A /\!/ \mathfrak{g} := A^{\mathfrak{g}} / J^{\mathfrak{g}}$ the quantum Hamiltonian reduction.

We can add FI parameters as following:

Let $\chi : \mathfrak{g} \to \mathbb{C}$ be a character of $\mathfrak{g}$, then define J_χ be left ideal generated by $\{\mu(a) - \chi(a)\cdot 1\}$

1 of A

then $J_\chi^{\mathfrak{g}}$ is 2-sided ideal of $A^{\mathfrak{g}}$,

so define $A /\!/_{\chi}\, \mathfrak{g} = A^{\mathfrak{g}} / J_\chi^{\mathfrak{g}}$

#

In our situation, $A = \mathbb{C}^{\varepsilon_1}[T^*(\mathrm{End}(\mathbb{C}^N) \oplus \mathrm{Hom}(\mathbb{C}^K, \mathbb{C}^N))][\varepsilon_2]$

A is generated by symbols $X_i^j \quad Y_k^l \quad I_i^a \quad J_b^j$

$\mathrm{End}(\mathbb{C}^N) \quad \mathrm{Hom}(\mathbb{C}^K, \mathbb{C}^N) \quad \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^K)$

with relations:

$$[X_i^j \; Y_k^l] = \varepsilon_1 \delta_i^l \delta_k^j, \quad [I_i^a \; J_b^j] = \varepsilon_1 \delta_i^j \delta_b^a$$

i, j, k, l $\mathfrak{gl}_N$

a, b $\mathfrak{gl}_K$

other Commutators are zero.

$$\mathfrak{g} = \mathfrak{gl}_N \qquad :[X, Y]_i^j: + (IJ)_i^j$$

$$\mu : t_i^j \longmapsto X_i^k Y_k^j - X_k^j Y_i^k + I_i^a J_a^j$$ F-term

Remark: The definition of QMM should be modified as

$$[\mu(a), b] = \varepsilon_1 \cdot \phi(a) \cdot b$$

$$\boxed{\begin{aligned} &\mu_{\varepsilon_2} = \mu - \varepsilon_2 \cdot \mathrm{tr}(\cdot) \\ &t_i^j \mapsto \mu(t_i^j) - \varepsilon_2 \cdot \delta_i^j \end{aligned}}$$

Taking q. Hamiltonian reduction, we get a $\mathbb{C}[\varepsilon_1, \varepsilon_2]$ algebra $\mathbb{C}^{\varepsilon_1}[\mathcal{M}_{N,K}^{\varepsilon_2}]$

Theorem $\mathbb{C}^{\varepsilon_1}[\mathcal{M}_{N,K}^{\varepsilon_2}]$ is flat over $\mathbb{C}[\varepsilon_1, \varepsilon_2]$

Sketch of proof: Introduce filtration on A by

$$\deg X = \deg Y = \deg I = \deg J = 1$$

$$\deg \varepsilon_1 = 0 \qquad \deg \varepsilon_2 = 0$$

Check that A is indeed filtered. $\mu(\mathfrak{gl}_N) \subset F_2 A$

Claim $$\operatorname{gr} \mathbb{C}^{\varepsilon_1}[\mathcal{M}_{N,K}^{\varepsilon_2}] \simeq \underbrace{\mathbb{C}[\mathcal{M}_{N,K}^{0}]}_{\text{Classical Ham. reduction}}[\varepsilon_1, \varepsilon_2]$$

It remains to prove the claim. Note that

$$\operatorname{gr} \mathbb{C}^{\varepsilon_1}[\mathcal{M}_{N,K}^{\varepsilon_2}] \simeq \operatorname{gr} A^{\mathfrak{gl}_N} / \operatorname{gr} J^{\mathfrak{gl}_N}$$

So the claim follows from the following:

Lemma: $\operatorname{gr} J$ is generated by $[X, Y]_i^{\ j} + I_i^{\ a} J_a^{\ j}$ as left ideal. $i, j \in \{1, \dots, N\}$

Sketch of proof. Let $E_i^{\hat{j}} = :[X,Y]_i^{\hat{j}}: + I_i^{a} J_a^{\hat{j}} - \varepsilon_2 \delta_i^{\hat{j}}$

it's enough to show that:

(*) If $\sum f_i^{\hat{j}} E_i^{\hat{j}} \in F_m A$, then $\exists \bar{f}_i^{\hat{j}} \in F_{m-2} A$

s.t. $\sum f_i^{\hat{j}} E_i^{\hat{j}} = \sum \bar{f}_i^{\hat{j}} E_i^{\hat{j}}$.

The claim (*) is a consequence of $\{E_i^{\hat{j}}\}_{i,j\in\{1,\cdots,N\}}$ is a regular sequence in $\mathrm{gr}\,A$, so we can subtract leading terms of $f_i^{\hat{j}}$ if they are not in $F_{m-2}A$, details omitted. #

Corollary. $\mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{N,k}] \simeq \mathcal{M}_H^{\varepsilon_1',\varepsilon_2'}$ (Higgs branch alg.)

$t_1,\ t$

for some ε_1' ε_2'

Proof: We see that $\mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{N,k}]$ and $\mathcal{M}_H^{\varepsilon_1'\varepsilon_2'}$ are filtered quantizations of conical symplectic singularity $\mathcal{M}^{0}_{N,k}$, $\varepsilon_2=0$

By the characterization theorem of filtered quantization (Ivan Losev), $\mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{N,k}] \simeq \mathcal{M}_H^{\varepsilon_1'\varepsilon_2'}$ for appropriate change of variables ε_1' ε_2'

#

Large-N Limit of $\mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{N,K}]$

This is subtle, since there is no embedding
$\mathcal{M}^{\varepsilon_2}_{N,K} \hookrightarrow \mathcal{M}^{\varepsilon_2}_{N+1,K}$ at classical level.
(Although there exists $\phi^N_{N+1} : A_{N+1} \twoheadrightarrow A_N$)

There is no naive $\varprojlim_N \mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{N,K}]$

Way Out: Study "Universal-in-N" instead.
See Costello's paper on M2 brane

Definition A sequence $\{f_N \in A_N\}$ is called admissible of weight 0 if ① f_N is GL_N-invariant

② $\phi^N_{N+1}(f_{N+1}) = f_N$

$\{f_N\}$ is called admissible of weight r if $\{N^{-r} f_N\}$ is admissible of weight 0. Example: $\{N^r\}$

$\{f_N\}$ is called admissible if it's a linear sum of admissible sequences of various weights.

Remark: $\{f_N\}, \{g_N\}$ admissible $\Rightarrow \{f_N g_N\}$ admissible.

Definition: Denote by $\mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{\bullet, K}]$ the algebra of admissible sequences modulo the ideal of adm. seq. in J_N

Lemma: $\mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{\bullet, K}]$ is generated over $\mathbb{C}[\varepsilon_1, \varepsilon_2]$ by

$$J_a{}^i (X^m Y^n)_i{}^j I_j{}^b, \quad \mathrm{Tr}(X^m Y^n), \quad \delta \text{ (central)}$$

weight: 0, 0, 1

In fact, $\delta = \{N\}$ is admissible of weight 1.

Theorem [Costello] Specialize ε_2 to a nonzero number and localize over $\mathbb{C}((\varepsilon_1))$, there is an isomorphism.

$$\mathbb{C}^{\varepsilon_1}(\mathcal{M}^{\varepsilon_2}_{\bullet, K}) \simeq \mathcal{U}_{\varepsilon_1}(D_{\varepsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K)\underline{[\varepsilon_1^{-1}]}$$

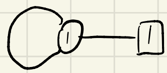
RHS: A deformation of Univ. enveloping alg. of Lie algebra

$D_{\varepsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K$

$\hookrightarrow$ Diff. operators on $\mathbb{C}$. $\left(\mathbb{C}\langle z, w \rangle / [z, w] = \varepsilon_2 \right)$

$K=1$ Case:

Coulomb Branch Point of View:

Example: $N=1$: 

$H^*_{gauge}(pt)$ $\hbar$

monopole operators $r_1 \quad r_{-1}$, equivariant parameter t, ε_1

relations:

$$r_1 r_{-1} = t \qquad r_{-1} r_1 = t - \varepsilon_1 \qquad [r_1, t] = \varepsilon_1 r_1 \qquad [r_{-1}, t] = -\varepsilon_1 r_{-1}$$

For general N, Kodera and Nakajima show that $\mathcal{M}_C^{\varepsilon_1}$ is isomorphic to Spherical part of graded Cherednik algebra $e H_N^{gr} e$, where H_N^{gr} is $\mathbb{C}[\varepsilon_1, \varepsilon_2]$ algebra generated by $S_1, \cdots, S_{N-1}$, $X_i^{\pm 1}$, w_i $(i=1, \cdots, N)$ w./ relations:

- $[w_i \; w_j] = [X_i \; X_j] = 0$
- $\overset{(12)}{S_1}, \overset{\cdots}{\cdots}, \overset{(N-1, N)}{S_{N-1}}$ generate $\mathbb{C}[S_N]$
- $S_i w_i = w_{i+1} S_i - \varepsilon_2 \qquad S_i w_{i+1} = w_i S_i + \varepsilon_2$

  $S_i w_j = w_j S_i$ otherwise
- $s X_i^{\pm 1} = X_{s(i)}^{\pm 1} s \qquad s \in S_N$

- $[X_j\ \omega_i] = \begin{cases} \varepsilon_2 X_j S_{ji} & \text{if } i>j \\ \varepsilon_2 X_i S_{ij} & \text{if } i<j \\ \varepsilon_1 X_i - \varepsilon_2 \sum\limits_{k<i} X_k S_{ki} - \varepsilon_2 \sum\limits_{k>i} X_i S_{ik} & \text{if } i=j \end{cases}$

Here $e = \frac{1}{N!} \sum\limits_{g \in S_N} g$

Kodera and Nakajima show that $\mathcal{M}_C^{\varepsilon_1}$ is quotient of $Y_1^{\varepsilon_1,\varepsilon_2}(\widehat{gl}_1)$ (1-shifted affine Yangian of gl_1)

Higgs Branch Point of View

Definition: let $\mathcal{A}$ be the $\mathbb{C}[\varepsilon_1\ \varepsilon_2]$ alg. generated by $\{t_{a,b}\}_{a,b\in\mathbb{Z}_{\geq 0}}^{\deg = a+b}$ w./ relations:

Fund $\otimes S^k \to S^{k-1}$ as sl_2-rep.

$t_{0,0}$ central

- $$\begin{cases} [t_{0,0}, t_{n,m}] = 0, \quad \boxed{[t_{1,0}, t_{n,m}] = m\, t_{n,m-1}, \quad [t_{0,1}, t_{n,m}] = n\, t_{n-1,m}} \\ \boxed{\begin{matrix} [t_{2,0}, t_{n,m}] = 2m\, t_{n+1,m-1}, & [t_{1,1}, t_{n,m}] = (m-n)\, t_{n,m} \\ [t_{0,2}, t_{n,m}] = -2n\, t_{n-1,m+1} & \end{matrix}} \end{cases}$$

$\bigoplus\limits_{n+m=k} t_{n,m}$ is spin k rep. of sl_2

↑ generate by $\{t_{2,0}, t_{1,1}, t_{0,2}\}$

Notation: $\sigma_2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_1\varepsilon_2$, $\sigma_3 = -\varepsilon_1\varepsilon_2(\varepsilon_1+\varepsilon_2)$

$$\bullet\ [t_{3,0}, t_{0,n}] = 3n\, t_{2,n-1} + \frac{3\sigma_2}{2}\binom{n}{3} t_{0,n-3} + \frac{3\sigma_3}{2}\sum_{m=0}^{n-3}(m+1)(n-2+m)\, t_{0,m}\, t_{0,n-3-m}$$

Remark: If we specialize to $\varepsilon_1 = \varepsilon_2 = 0$, then

$$A_{\varepsilon_1=\varepsilon_2=0} \simeq \mathcal{U}\Big(\underbrace{\{t_{a,b} \mid [t_{a,b}, t_{c,d}] = (ad-bc)\, t_{a+c-1,\, b+d-1}\}}\Big)$$

Lie algebra of functions on $\mathbb{C}^2_{z,w}$ with Poisson bracket $\{z,w\}=1$

$t_{n,m} \mapsto z^n w^m$

Theorem [PBW-type] $\mathcal{A}$ is a free $\mathbb{C}[\varepsilon_1, \varepsilon_2]$-module with basis

$$t_{a_1 b_1} \cdots t_{a_n b_n} \quad \text{such that} \quad (a_1\, b_1) \leq \cdots \leq (a_n\, b_n)$$

where "$\leq$" is a total order on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

Proposition: There exists $\mathbb{C}[\varepsilon_1^{\pm 1}, \varepsilon_2]$-algebra homomorphism

$$\mathcal{A}[\varepsilon_1^{-1}] \longrightarrow \mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{\bullet,1}][\varepsilon_1^{-1}]$$

$(n,m) \neq (0,0)$ $\quad t_{n,m} \longmapsto \frac{1}{\varepsilon_1}\,\mathrm{SymTr}(X^n Y^m)$

$$t_{0,0} \longmapsto \frac{1}{\varepsilon_1}\cdot \delta$$

The proof is by direct computation.

Consider subalgebra $\mathcal{B} \subset \mathcal{A}$ generated by $t'_{n,m} = \varepsilon_1 \cdot t_{n,m}$

it's easy to see that $\mathcal{B}$ is free over $\mathbb{C}[\varepsilon_1, \varepsilon_2]$ w./ basis

$$t'_{a_1 b_1} \cdots t'_{a_n b_n} \qquad (a_1\ b_1) \leq \cdots \leq (a_n\ b_n)$$

And $B/(\varepsilon_1\ \varepsilon_2)B = \mathrm{Sym}^\bullet(\{t'_{n,m}\}_{n,m \in \mathbb{Z}_{\geq 0}})$

Moreover the map $\mathcal{A}[\varepsilon_1^{-1}] \longrightarrow \mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{\bullet,1}][\varepsilon_1^{-1}]$ sends $\mathcal{B}$ to $\mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{\bullet,1}]$

Theorem: The homomorphism $\mathcal{B} \longrightarrow \mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{\bullet,1}]$ is isomorphism

Sketch of proof: $\mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{\bullet,1}]$ is generated by $\mathrm{Sym}\,\mathrm{Tr}(X^n Y^m)$ and δ, since (use F-term)

$$J^i \mathrm{Sym}(X^n Y^m)_i{}^j I_j = \varepsilon_2 \cdot \mathrm{Sym}\,\mathrm{Tr}(X^n Y^m)$$

Thus $\mathcal{B} \to \mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{\bullet,1}]$ is surjective $t'_{n,m} \mapsto S\mathrm{Tr}(X^n Y^m)$

$$t'_{0,0} \mapsto \delta$$

It remains to show injectivity. We claim that

$$\mathcal{B}/(\varepsilon_1\ \varepsilon_2) \longrightarrow \mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{\bullet,1}]/(\varepsilon_1\ \varepsilon_2)$$

is injective.

Observation: Both sides are graded and the map preserves the grading.

$$\deg t_{n,m} = n+m \quad \mapsto ST_r(x^n y^m)$$
$$\deg X = \deg Y = \deg I = \deg J = 1$$
$$\deg \delta = 0$$

What is $\mathbb{C}[\mathcal{M}^\circ_{\bullet,1}]$? $\qquad \mathcal{M}^\circ_{N,1} = S^N(\mathbb{C}^2)$.

thus weight 0 admissible sequences is the subalgebra of $\varprojlim_N \mathbb{C}[S^N(\mathbb{C}^2)]$ generated by $\mathbb{C}^\times$-eigen vectors $STr(x^n y^m)$ where $\mathbb{C}^\times \circlearrowright \mathbb{C}^2$ w./ weights $-1, -1$.

This is known as "Ring of MacMahon Symmetric Functions"

Denote it by S.

Theorem [See "Multisymmetric Functions" J. Dalbec 1999]

$$S \simeq \mathbb{C}\left[M_{(a,b)} \;\middle|\; (a,b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \setminus (0,0)\right]$$

where $M_{(a,b)}$'s image in $\mathbb{C}[S^N(\mathbb{C}^2)]$ is

$$x_1^a y_1^b + x_2^a y_2^b + \cdots + x_N^a y_N^b$$

(Analog of power sum in $S^N(\mathbb{C})$)

$$P_a = x_1^a + x_2^a + \cdots + x_N^a$$

It is easy to see that δ has no algebraic relations with $m_{(a,b)}$, thus $\mathbb{C}[\mathcal{M}^{\circ}_{\bullet,1}] \simeq S[\delta]$

On the othe hand,

$$B/(\varepsilon_1, \varepsilon_2) \simeq \underbrace{\mathrm{Sym}\left(\{t_{n,m} \mid (n,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \setminus (0,0)\}\right)}_{\text{Has the same } \mathbb{C}^{\times}\text{-weights with } S}[t_{0,0}]$$

Therefore $B/(\varepsilon_1, \varepsilon_2) \xrightarrow{\sim} \mathbb{C}[\mathcal{M}^{\circ}_{\bullet,1}]$

The injectivity before modulo $(\varepsilon_1, \varepsilon_2)$ is deduced from the above and the Leading Term Trick

Details omitted #

Corollary: The homomorphism $\mathcal{A}[\varepsilon_1^{-1}] \to \mathbb{C}^{\varepsilon_1}[\mathcal{M}^{\varepsilon_2}_{\bullet,1}][\varepsilon_1^{-1}]$ is isomorphism

Remark: Compare our $\mathcal{A}$ with N. Guay, Etingof's deformed double current algebra (DDCA), we see that they are isomorphic, see Proposition 4.2.13 of 2005.13604

This confirms one of Kevin's Conjecture in the case $K=1$.

Coproduct of $\mathcal{A}$:

Consider the map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}((z^{-1}))$

$$\begin{cases} \Delta(t_{0,n}) = 1 \otimes t_{0,n} + \sum_{m=0}^{n} \binom{n}{m} z^{n-m} t_{0,m} \otimes 1 \\ \Delta(t_{2,0}) = 1 \otimes t_{2,0} + t_{2,0} \otimes 1 + 2\sigma_3 \sum_{m,n \geq 0} \frac{(m+n+1)!}{m!\, n!} (-1)^n z^{-n-m-2} t_{0,n} \otimes t_{0,m} \end{cases}$$

Proposition [Gaiotto-Rapčák]

Δ extends to an algebra homomorphism.

In fact, Δ has more structures, to explain it, we need some notations.

Definition A Vertex Operator Coalgebra (VOC) is a vector space V together w/ linear maps.

- [Coproduct] $\curlywedge(z) : V \to (V \otimes V)((z^{-1}))$
- [Covacuum] $c : V \to \mathbb{C}$

"Dualize all ingredients of VOA"

Satisfying axioms:

① Left Counit : $\forall v \in V$

$$(C \otimes Id_V)\,\lambda(z)\cdot v = v$$

② Cocreation : $\forall v \in V$

$$(Id_V \otimes C)\,\lambda(z)\,v \in V[z] \quad \text{and}$$

$$\lim_{z \to 0} (Id_V \otimes C)\,\lambda(z)\,v = v$$

③ Jacobi Identity : $\forall v \in V$

$$z_0^{-1}\,\delta\left(\frac{z_1 - z_2}{z_0}\right)\left(Id_V \otimes \lambda(z_2)\right)\lambda(z_1)$$

$$- z_0^{-1}\,\delta\left(\frac{z_2 - z_1}{-z_0}\right)\left(T \otimes \lambda(z_1)\right)\lambda(z_2)$$

$$= z_2^{-1}\,\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\lambda(z_0) \otimes Id_V\right)\lambda(z_2)$$

where $\delta\left(\frac{y - x}{z}\right) = \sum_{m \geq 0,\ n \in \mathbb{Z}} \binom{n}{m} y^{n-m} x^m z^{-n}$

and $T : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_2$ swaps two components

Proposition Consider linear map $c : A \to \mathbb{C}$ sends 1 to 1 and $t_{n,m}$ to 0, then (A, Δ, c) is a UOC.

Proof is by direct computation.